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The Mighty Maximin

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Abstract

In the framework of classical decision theory, the quest for robustness against severe uncertainty is almost synonymous with the use of Wald's famous – some would say notorious – Maximin paradigm, or with one of its many variants. In this discussion we examine certain modeling issues that are pertinent to this stalwart of classical decision theory, with a view to show that the austere simplicity of its mathematical structure is often mistaken for rigidity.

We illustrate this point by examining the modeling of this paradigm in the context of "robust satisficing" problems. That is, we examine the paradigm's application to problems where robustness is sought with respect to performance constraints rather than the value of the objective function.

By this I do not mean to suggest that I am advocating Maximin as a panacea for decision under severe uncertainty. Therefore, one should not expect to find in this article lengthy commentary on the suitability of this paradigm for the treatment of severe uncertainty.

Rather, my main goal is to point out that notwithstanding the austere mathematical idiom in which it is formulated, the Maximin paradigm puts at our disposal a powerful tool of thought that proves extremely handy in modeling.

This discussion has been motivated by the prevalence of various misconceptions about Maximin in the Info-Gap literature.

Keywords: Maximin, uncertainty, variability, robustness, satisficing, Info-Gap.

1 Introduction

Over the past four years I have had numerous occasions to discuss the notion of "robustness" and its various interpretations in the framework of decision-making under severe uncertainty, with academics, students, and practitioners.

To my great surprise, I discovered that Wald's Maximin paradigm – the most prominent paradigm in classical decision theory and robust optimization for the treatment of severe uncertainty – has a serious "public relations" problem.

For one thing, it is quite obvious that this stalwart of classical decision theory and robust optimization is not as known as it should be to researchers/analysts working in the area of risk analysis. But what is more, some of those who are familiar with it have serious misconceptions about its scope of operation, its modeling capabilities, and its role and place in robust decision-making.

So, this article is the product of my ongoing effort to improve the public image of Maximin among researchers/analysts in the area of decision-making under severe uncertainty whose knowledge of Maximin is limited.

I should make it clear, though, that I am not a Maximin fanatic, or an aficionado, nor do I seek to convert readers to Maximin.

Rather, my aim is to increase awareness of Maximin's mode of operation, its role and place in decision theory and so on. So, my focus in this article is on the modeling aspects of Maximin, more specifically, on what I call Maximin models in disguise. And to do this, I shall concentrate exclusively on its application in situations of severe uncertainty rather than on situations of known variability.

2 First encounter

On the face of it, the Maximin paradigm seems to be something of a second nature to many of us: its basic attitude to uncertainty is summed up in the familiar, indeed widely held, maxims:

When in doubt, assume the worst!
Hope for the best and prepare for the worst![1]

For instance, the following quote is taken from the *Vegetarian Society of Ireland*[2] Website:

> EDIBLE FAT / FAT
> could be animal or vegetable. If in doubt, assume the worst!

And this is a quote from the RevolutionHealth.com[3] website:

> Young children have the highest risk of poisoning because of their natural curiosity. More than half of poisonings in children occur in those who

[1] According to WIKIPEDIA's list of quotes, this traditional proverb is found in Roger L'Estrange, Seneca's Morals (1702).

[2] http://www.vegetarian.ie/productpage.htm

[3] http://www.revolutionhealth.com/conditions/first-aid-safety/first-aid-treatment/poisoning/emergencies

> are younger than age 6. Some children will swallow just about anything, including unappetizing substances that are poisonous. When in doubt, assume the worst. Always believe a child or a witness, such as another child or a brother or sister, who reports that poison has been swallowed. Many poisonings occur when an adult who is using a poisonous product around children becomes distracted by the doorbell, a telephone, or some other interruption.

Next, consider a quote taken from the defense/military literature, more specifically from the Janes.Com website[4]:

> Opinion: Let sleeping mines lie?
> By Colin King Editor of Jane's Mines and Mine Clearance and Jane's Explosive Ordnance
>
> The deeply entrenched attitude of explosive ordnance disposal (EOD) personnel to safety means that deteriorated ammunition is always presumed to be less stable than serviceable ordnance. This view is reinforced by a few examples (such as the highly sensitive compounds formed by reactions with certain early high explosives) and the EOD adage that, if in doubt, assume the worst.

Or, consider the following table in the paper "Preventing Errors in Clinical Practice: A Call for Self-Awareness"[5]

Table 3. Examples of Low-Level vs High-Level Schemata

Low-Level Schemata	High-Level Schemata
I've got it! As soon as the patient told me, I knew what he had.	I should look beyond early hypotheses.
If the patient is satisfied with the diagnosis of another physician, why should I bother to find out more data?	I should always form my own criteria.
When in doubt, choose the simplest or most convenient hypothesis.	When in doubt, assume the worst hypothesis.
Complains a lot? He doesn't have anything!	I must take a fresh look - perhaps by recording what the patient expresses and later reading it back, paying attention to the diagnosis that spontaneously comes to mind.

And here is some practical advice on the behavior of sharks from the compilation entitled "Shark Diving For Dummies"[6]:

> The difference between curiosity and animosity is subtle. When in doubt, assume the worst and leave the water.

[4]http://www.janes.com/defence/news/jdw/jdw060904_2_n.shtml

[5]Francesc Borrell-Carrió and Ronald M. Epstein, ANNALS OF FAMILY MEDICINE, 2(4), 310-316, 2004.

[6]See http://www.elasmodiver.com/Shark_Diving_For_Dummies.htm. First published in *Xray Magazine*, #34 (Jan 2010)

And finally, the most venerable of them all:

> The gods to-day stand friendly, that we may,
> Lovers of peace, lead on our days to age!
> But, since the affairs of men rests still incertain,
> Let's reason with the worst that may befall.

Julius Caesar, Act 5, Scene 1
William Shakespeare (1564-1616)

The point is then that the Maximin paradigm is a decision-making methodology that is based on a worst-case scenario.

For those who have not encountered this paradigm, here is how the philosopher John Rawls (1971, p. 152) formulates it in the discussion of his theory of justice:

> The maximin rule tells us to rank alternatives by their worst possible outcomes: we are to adopt the alternative the worst outcome of which is superior to the worst outcome of the others.

The careful reader may no doubt observe that the term "superior" is too demanding here. Hence, consider the following slightly modified version of Rawls' formulation:

Maximin Maxim

Rank alternatives by their worst possible outcomes: adopt the alternative the worst outcome of which is at least as good as the worst outcome of the others.

In classical decision theory (Resnik 1987, French 1998) this paradigm has become the standard model for dealing with severe uncertainty. This has been the case ever since Wald (1945, 1950) adapted von-Neumann's (1928, 1944) Maximin paradigm for game theory by casting "uncertainty", or "Nature", as one of the two players. Here is a simple example of the Maximin in action in such a framework.

Consider the following *decision table* where the rows represent alternatives (decisions), the columns represent the uncertain states (of nature) and the entries represent the outcomes, say the number of dark chocolate bars you gain.

	s_1	s_2	s_3	s_4
d_1	3	4	2	6
d_2	3	49	1	83
d_1	3	4	3	4

For the purpose of this exercise assume that you love dark chocolate, so the more the better.

To decide what is the best choice á la Maximin, we append a column to the decision table in which we enter the worst outcome for each decision. Here is what we obtain:

	s_1	s_2	s_3	s_4	SL
d_1	3	4	2	6	2
d_2	3	49	1	83	1
d_3	3	4	3	4	3

where SL is short for Security Level.

So the best worst-case outcome in this case is the one associated with decision d_3, namely 3 tasty dark chocolate bars.

Note that this outcome: the gain of 3 dark chocolate bars, is extremely robust considering the uncertainty in the value of the state s. In fact, it is the most robust one can obtain: regardless of how bad the state s will be, one is assured of at least 3 bars.

In the parlance of classical decision theory, the worst outcome associated with a decision is the *security level* of the decision.

In short then, in the framework of the Maximin paradigm, the best decision is that whose security level is the greatest.

It is clear from what we have seen thus far that the attractiveness of the Maximin paradigm is in its apparent ability to dissolve the uncertainty associated with Nature's selection of its states. This is due to the underlying assumption that nature is a consistent adversary. That is, playing against the decision-maker (DM), Nature consistently selects the least favorable state associated with the decision selected by the DM. This has the effect of allowing the DM to predict Nature's moves to thereby eliminate the uncertainty from the analysis.

The price tag attached to this convenience is significant: by completely removing the uncertainty and focusing exclusively on the worst outcome, the Maximin may yield highly "conservative" outcomes (Tintner 1952). It is not surprising, therefore, that over the years a number of attempts have been made to modify this paradigm with a view to mitigate its extremely "pessimistic" stance. The most famous variation is no doubt Savage's Minimax Regret model (Savage 1951, Resnik 1987, French 1988). The fact remains though, that for all this effort, the Maximin paradigm provides no easy remedy to handle decision problems subject to severe uncertainty/variability (Harsanyi 1976).

3 Math formulations

By way of introduction, it should be noted that there is more then one way to formulate the Maximin paradigm mathematically. For our purposes, however, it will suffice to consider two popular equivalent formulations, namely the classical formulation and the mathematical programming formulation. As we shall see, these formulations can often be simplified by exploiting specific features of the problem under consideration.

Both formulations are based on the following three basic, simple, intuitive, abstract constructs:

- A *decision space,* D.
  A set consisting of all the decisions available to the decision maker.
- *State spaces,* $S(d) \subseteq \mathbb{S}, d \in D$.
  $S(d)$ represents the set of states associated with decision $d \in D$. We refer to $\mathbb{S}$ as the *state space.*
- A real-valued function f on $D \times \mathbb{S}$.
  $f(d, s)$ represents the value of the outcome generated by the decision-state pair

(d, s). We refer to f as the *objective function.*

The decision situation represented by this model is as follows: the DM is intent on selecting a decision that will optimize the value generated by the objective function f. However, this value depends not only on the decision d selected by the DM, but also on the state s selected by Nature.

Since Nature is a consistent adversary, it will always select a state $s \in S(d)$ that is least favorable to the DM. Thus, if the DM is maximizing, Nature will minimize $f(d, s)$ with respect to s over $S(d)$. And if DM is minimizing, Nature will maximize $f(d, s)$ with respect to s over $S(d)$

3.1 Classical formulation

This formulation has two forms, depending on whether the DM seeks to maximize or minimize the objective function:

$$\text{Maximin Model}: \quad z^* = \max_{d \in D} \min_{s \in S(d)} f(d, s) \tag{1}$$

$$\text{Minimax Model}: \quad z^* = \min_{d \in D} \max_{s \in S(d)} f(d, s) \tag{2}$$

Note that in these formulations the "outer" optimization represents the DM and the "inner" optimization represents Nature. This means that the DM "plays" first and Nature's response is contingent on the decision selected by the DM.

3.2 Mathematical programming formulation

Often it proves more convenient to express the above models as a "conventional" optimization models by eliminating the "inner" optimization altogether. Here are two equivalent models resulting from such a re-formulation:

$$\begin{array}{ll|ll} \multicolumn{2}{c}{\text{Maximin Model}} & \multicolumn{2}{c}{\text{Minimax Model}} \\ \hline z^* := & \max\limits_{\substack{d \in D \\ v \in \mathbb{R}}} v & z^* := & \min\limits_{\substack{d \in D \\ v \in \mathbb{R}}} v \\ \text{s.t} & v \leq f(d, s) \ , \ \forall s \in S(d) & \text{s.t} & v \geq f(d, s) \ , \ \forall s \in S(d) \end{array} \tag{3}$$

where $\mathbb{R}$ denotes the real-line.

The "disappearance" of the inner optimization operations is no cause for concern, observing that these models can also be written as follows:

$$\begin{array}{ll|ll} \multicolumn{2}{c}{\text{Maximin Model}} & \multicolumn{2}{c}{\text{Minimax Model}} \\ \hline z^* := & \max\limits_{\substack{d \in D \\ v \in \mathbb{R}}} v & z^* := & \min\limits_{\substack{d \in D \\ v \in \mathbb{R}}} v \\ \text{s.t} & v \leq \min\limits_{s \in S(d)} f(d, s) & \text{s.t} & v \geq \max\limits_{s \in S(d)} f(d, s) \end{array} \tag{4}$$

For our purposes it would be most convenient to formulate these models as follows:

$$\text{Maximin Model} \quad z^* := \max_{\substack{d \in D \\ v \in \mathbb{R}}} \{v : v \leq f(d, s), \forall s \in S(d)\} \tag{5}$$

$$\text{Minimax Model} \quad z^* := \min_{\substack{d \in D \\ v \in \mathbb{R}}} \{v : v \geq f(d, s), \forall s \in S(d)\} \tag{6}$$

Note that in all these formulations v is a decision variable.

The $\forall s \in S(d)$ in the constraint entails that in cases where the state spaces are "continuous" rather than discrete, the Maximin model represents a semi-infinite optimization problem (Reemstem and Rückmann 1998).

Now, since the Minimax model and the Maximin model are equivalent (via the multiplication of the objective function by -1), we shall henceforth concentrate only on the Maximin model.

4 Robustness

To be clear on what is meant by robustness, consider the first paragraph of the entry *Robustness* in WIKIPEDIA:

> **Robustness** *is the quality of being able to withstand stresses, pressures, or changes in procedure or circumstance. A system, organism or design may be said to be "robust" if it is capable of coping well with variations (sometimes unpredictable variations) in its operating environment with minimal damage, alteration or loss of functionality.*

Hence, in this vein, we say that a decision is robust if its outcomes or consequences are capable of coping well with changes and variations in the decision-making environment under consideration.

In this discussion it is instructive to distinguish between the following three generic types of robustness:

- Robust satisficing.
  Robustness is sought with respect to the constraints associated with a satisficing problem or an optimization problem.
- Robust optimizing.
  Robustness is sought with respect to the objective function of an optimization problem.
- Robust optimizing and satisficing.
  Robustness is sought with respect to both the objective function and constraints associated with an optimization problem.

Note that the Maximin formulations discussed above deal with robust optimizing problems. The question is then: how does the Maximin paradigm deal with robust satisficing and robust optimizing and satisficing problems? What are the mathematical formulations of Maximin models representing such problems?

5 Robust satisficing Maximin models

To facilitate the application of the Maximin model in the context of robust satisficing rather than in that of robust optimizing problems, one can appeal to a number of simple modeling devices.

And to illustrate, consider the following simple robust satisficing model:

$$\text{Find an } x \in X \text{ such that } g(x,u) \leq b \text{ and } h(x,u) \geq c, \; \forall u \in \mathfrak{U} \tag{7}$$

where X and $\mathfrak{U}$ are some given sets, b and c are given numeric constants, and g and h are real-valued functions on $X \times \mathfrak{U}$.

What makes this model a Maximin model par excellence is the clause $\forall u \in \mathfrak{U}$. That is, an alternative $x \in X$ is evaluated on the basis on its worst performance in relation to the requirement $u \in \mathfrak{U}$. Note that no more than a single $u \in \mathfrak{U}$ is required to render x "unacceptable" because this "bad" u – in conjunction with x – violates one or more of the constraints.

To formally phrase this robust satisficing model as a "classical" Maximin model, let

$$\varphi(x,u) = \begin{cases} 1 & , \; g(x,u) \leq b \text{ and } h(x,u) \geq c \\ 0 & , \; \text{otherwise} \end{cases} \; , \; x \in X, u \in \mathfrak{U} \tag{8}$$

Then, clearly, by inspection, $x^* \in X$ is a solution to the robust satisficing model iff it is an optimal solution to the following Maximin model:

$$z^* := \max_{x \in X} \min_{u \in \mathfrak{U}} \varphi(x,u) \tag{9}$$

The equivalent mathematical programming formulation of this Maximin model is as follows:

$$z^* := \max_{\substack{x \in X \\ v \in \{0,1\}}} \; \{v : v \leq \varphi(x,u), \forall u \in \mathfrak{U}\} \tag{10}$$

Now, consider the following more complicated robust satisficing model:

$$\max_{x \in X} \; \beta(x) \tag{11}$$

$$\text{s.t.} \begin{cases} g(x,u) \leq b \\ h(x,u) \geq c \end{cases} \; , \; \forall u \in \mathfrak{U} \tag{12}$$

where X and $\mathfrak{U}$ are some given sets, b and c are given numeric constants, β is a real-valued function on X, and g and h are real-valued functions on $X \times \mathfrak{U}$.

Is this a Maximin model?

Of course it is!

The worst $u \in \mathfrak{U}$ with respect to a given $x \in X$ is that which violates the constraint (12). If none of the elements of $\mathfrak{U}$ violates the constraint then x is a robust decision with respect to (12). So the model requires that we find a robust decision that maximizes the objective function β over X.

The classical formulation of the equivalent Maximin model is as follows:

$$z^* := \max_{x \in X} \min_{u \in \mathfrak{U}} \; \psi(x,u) \tag{13}$$

where

$$\psi(x,u) = \begin{cases} \beta(x) & , \quad g(x,u) \leq b \text{ and } h(x,u) \geq c \\ -\infty & , \quad \text{otherwise} \end{cases} \quad , \ x \in X, u \in \mathfrak{U} \tag{14}$$

and, as usual, if $z^* = -\infty$, then the implication is that the problem represented by the model has no feasible solution.

Note that the corresponding mathematical programming formulation of the Maximin model is as follows:

$$\begin{aligned} z^* := & \max_{\substack{x \in X \\ v \in \mathbb{R}}} \{v : v \leq \psi(x,u), \forall u \in \mathfrak{U}\} & (15) \\ = & \max_{x \in X} \{\beta(x) : \beta(x) \leq \psi(x,u), \forall u \in \mathfrak{U}\} & (16) \\ = & \max_{x \in X} \{\beta(x) : g(x,u) \leq b, h(x,u) \geq c, \forall u \in \mathfrak{U}\} & (17) \end{aligned}$$

confirming the fact that (11)-(12) is indeed a Maximin model.

6 Robust optimizing and satisficing

Consider the following generic parametric optimization model:

$$\text{Problem } P(u), u \in \mathfrak{U}: \quad z^*(u) := \max_{x \in X(u)} g(x,u) \tag{18}$$

where $\mathfrak{U}$ is a given set, $X(u) \subseteq \mathfrak{X}, \forall u \in \mathfrak{U}$ for some given set $\mathfrak{X}$, and g is a real-valued function on $X \times \mathfrak{U}$.

Observe that here both the solution set, $X(u)$, and the objective function, g, are contingent on the parameter u. Let $X^*(u)$ denote the set of optimal solutions to *Problem* $P(u)$ for $u \in \mathfrak{U}$.

If there exists an $x^* \in \mathfrak{X}$ such that $x^* \in X^*(u), \forall u \in \mathfrak{U}$ then we can safely proclaim x^* to be super-robust. This solution is obviously robust satisficing, namely

$$x^* \in \mathcal{X} := \bigcap_{u \in \mathfrak{U}} X(u) \tag{19}$$

But more than that: for every $u \in \mathfrak{U}$ it maximizes the objective function g over $X(u)$, namely

$$x^* = \arg \max_{x \in X(u)} g(x,u) \ , \ \forall u \in \mathfrak{U} \tag{20}$$

The trouble is, of course, that super-robust solutions are rare events. Indeed, in cases where $g(x,u)$ and $X(u)$ depend on u in a non-trivial manner, the existence of a super-robust solution for a robsut-optimizing and satisficing problem is unlikely.

The question is then: how should we define robustness with respect to the objective function g in the framework of *Problem* $P(u)$?

The following Maximin model is an obvious option, but decidedly not the only one:

$$z^* := \max_{x\in\mathcal{X}} \min_{u\in\mathfrak{U}} \ g(x,u) \tag{21}$$

An optimal solution to the problem represented by this model is a robust satisficing solution whose worst outcome with respect to g is at least as good as the worst outcome of any other robust satisficing solution.

7 Partial robustness

Since the introduction of Maximin the consensus in decision theory has been that the Maximin paradigm is too conservative (Tintner 1952). Specifically, the criticism has been directed at the use of the worst-case approach as a means of ranking the performance of decisions.

The argument is that by singling out the worst case as a measure of performance, the Maximin paradigm takes an excessively grim (pessimistic) view of uncertainty. This point is eloquently summarized by French (1988, p. 37):

> It is, perhaps, a telling argument against Wald's criterion that, although there are many advocates of this approach, there are few, if any, of the maximax return criterion. Why is it more rational to be pessimistic than optimistic? An old proverb may tell us that 'it is better to be safe than sorry', and it is true that Wald's criterion is as cautious as possible: but one must remember also that 'nothing ventured, nothing gained'.

Classical decision theory provides two adaptations of the Maximin paradigm to deal with this issue (Resnik 1987, French 1988):

- Savage's Minimax regret paradigm:

$$z^* := \min_{d\in D} \max_{s\in S(d)} r(d,s) \tag{22}$$

  where

$$r(d,s) := \left\{\max_{d\in D} f(d,s)\right\} - f(d,s) \tag{23}$$

- Hurwicz's optimism-pessimism paradigm:

$$z^*(\alpha) := \max_{d\in D}\left\{\alpha \min_{s\in S(d)} f(d,s) + (1-\alpha) \max_{s\in S(d)} f(d,s)\right\} \ , \ \alpha\in[0,1] \tag{24}$$

  where α, the famous optimisim-pessimism index, measures how optimistic/pessimistic the decision maker is with regard to the uncertain state of nature.

Unfortunately, the need for such adaptations is often misinterpreted as an indication that the Maximin paradigm per se disallows control of its conservatism.

It is important to point out, therefore, that as such the Maximin's conservatism is not set in stone, meaning that there are definitely ways to control it. This is particularly pertinent to robustness. The point to note here is that standard modeling tools can be used to *control* the degree, namely severity, of the robustness provided by the Maximin model. It is therefore up to the modeler/analyst to decide how conservative/liberal a particular robustness model á la Maximin should be with respect to the objective function and the constraints.

For instance, suppose that instead of insisting that a robust decision $d \in D$ should perform well over the entire state space $S(d)$ associated with it, we only require d to perform well on a large subset of $S(d)$. In fact, how about defining the robustness of d as the "size" of the largest subset of $S(d)$ over which d performs well? Better yet, how about expressing the "size" as a percentage of the total region of uncertainty? We could then say, for instance, that a certain decision is robust over 84% of its respective region of uncertainty?

Irrespective of how we shall ultimately define the "size" of the set over which a decision performs well, we refer to robustness of this kind as partial robustness.

As we show next, the Maximin paradigm provides a suitable modeling framework for partial robustness.

7.1 Partial robust satisficing

Consider the following generic robust satisficing model:

$$\textit{Find an } x \in X \textit{ such that } h(x,u) \in C, \forall u \in \mathfrak{U} \tag{25}$$

where X, C and $\mathfrak{U}$ are given sets and $h : X \times \mathfrak{U} \to C$.

Suppose that this model has no feasible solution because the robustness requirement $h(x,u) \in C, \forall u \in \mathfrak{U}$ is too exacting.

In this case we may have to make do with obtaining partial robustness. That is, we shall drop the requirement that the constraint $h(x,u) \in C$ be satisfied for all $u \in \mathfrak{U}$. Instead, we shall require that the constraint be satisfied only on a subset of $\mathfrak{U}$. It goes without saying that – under severe uncertainty – we shall seek to make this subset as large as possible.

In other words, the idea is to find a decision that is robust over the largest subset of $\mathfrak{U}$.

To this end, let $\rho(U)$ denote the "size" of set $U \subseteq \mathfrak{U}$. Formally, view ρ as a real-valued function on the power set of $\mathfrak{U}$ such that

$$\rho(U) \geq 0 \ , \ \forall U \subseteq \mathfrak{U} \tag{26}$$

$$U \subset U' \longrightarrow \rho(U) < \rho(U') \tag{27}$$

We now consider the following much less demanding formulation of the generic robust satisficing model (25):

$$r^* := \max_{\substack{x \in X \\ U \subseteq \mathfrak{U}}} \ \rho(U) \tag{28}$$

$$h(x,u) \in C \ , \ \forall u \in U \tag{29}$$

In words, we seek a decision $x \in X$ that maximizes the size of the subset of $\mathfrak{U}$ over which the constraint $h(x,u) \in C$ is satisfied for each u in this subset.

Again, the $\forall u \in U$ clause indicates that this is a Maximin model. Indeed, here is the classical formulation of the model:

$$r^* := \max_{\substack{x \in X \\ U \subseteq \mathfrak{U}}} \min_{u \in U} \ \sigma(x,U,u) \tag{30}$$

where

$$\sigma(x,U,u) := \begin{cases} \rho(U) & , \quad h(x,u) \in C \\ -\infty & , \quad h(x,u) \notin C \end{cases} , \ x \in X, U \subseteq \mathfrak{U}, u \in U \tag{31}$$

The mathematical programming formulation of this model is as follows:

$$\begin{align} r^* := & \max_{\substack{x \in X \\ U \subseteq \mathfrak{U} \\ v \in \mathbb{R}}} \{ v : v \leq \sigma(x,U,u), \forall u \in U \} \tag{32} \\ = & \max_{\substack{x \in X \\ U \subseteq \mathfrak{U}}} \{ \rho(U) : \rho(U) \leq \sigma(x,U,u), \forall u \in U \} \tag{33} \\ = & \max_{\substack{x \in X \\ U \subseteq \mathfrak{U}}} \{ \rho(U) : h(x,u) \in C, \forall u \in U \} \tag{34} \end{align}$$

which, needless to say, is equivalent to (28)-(29).

The trouble with this mighty model is that unless the topology of $\mathfrak{U}$ is simple, the optimization problem (28)-(29), hence (34) could be extremely difficult to solve.

7.2 Partial robust optimizing

Along the same lines, in the case of robust optimizing problems we can define robustness as the size of the largest subset of the total region of uncertainty over which the objective function attains a "satisfactory" value. For example, consider the following generalization of the above model:

$$r^* := \max_{\substack{x \in X \\ U \subseteq \mathfrak{U}}} \ \rho(U) \tag{35}$$

$$f(x,u) \geq f^*(u) - \varepsilon \ , \ \forall u \in U \tag{36}$$

where $\varepsilon \geq 0$ and

$$f^*(u) := \max_{x \in X} f(x,u) \ , \ u \in \mathfrak{U} \tag{37}$$

and f denotes the objective function under consideration.

In words, we are seeking a decision $x^* \in X$ that yields the largest subset U of $\mathfrak{U}$ over which x^* is ε-optimal.

We note that this model is an abstraction/generalization of Star's (1963, 1966) somewhat neglected Domain Criterion (Schneller and Sphicas 1983, Eiselt and Langley 1990, Eiselt et al 1998, Lempert and Collins 2007). The interesting thing is that this model is in fact a Maximin model. Indeed, here is the classical Maximin formulation of this model:

$$r^* := \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \min_{u\in U} \ \vartheta(x,U,u) \tag{38}$$

where

$$\vartheta(x,U,u) := \begin{cases} \rho(U) & , \ f(x,u) \geq f^*(u) - \varepsilon \\ -\infty & , \ \text{otherwise} \end{cases} \ , \ x\in X, U\subseteq \mathfrak{U}, u\in U \tag{39}$$

The equivalent mathematical programming formulation is then as follows:

$$r^* : = \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}\\ v\in\mathbb{R}}} \ \{v : v \leq \vartheta(x,U,u), \forall u\in U\} \tag{40}$$

$$= \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \ \{\rho(U) : \rho(U) \leq \vartheta(x,U,u), \forall u\in U\} \tag{41}$$

$$= \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \ \{\rho(U) : f(x,u) \geq f^*(u) - \varepsilon, \forall u\in U\} \tag{42}$$

As in the case of complete robustness, unless the topology of $\mathfrak{U}$ is simple, the optimization problem could be extremely difficult to solve[7].

We remark in passing that in this framework the robustness of decision $x \in X$ is prescribed as follows:

$$r(x) := \max_{U\subseteq \mathfrak{U}} \ \rho(U) \tag{43}$$

$$f(x,u) \geq f^*(u) - \varepsilon \ , \ \forall u\in U \tag{44}$$

This in itself is a Maximin model, namely

$$r(x) := \max_{U\subseteq \mathfrak{U}} \{\rho(U) : f(x,u) \geq f^*(u) - \varepsilon \ , \ \forall u\in U\} \equiv \max_{U\subseteq \mathfrak{U}} \ \min_{u\in U} \vartheta(x,U,u) \tag{45}$$

7.3 Partial robust optimizing and satisficing

Along the same lines, in the case of robust optimizing and satisficing problems we can define robustness as the size of the largest subset of the total region of uncertainty over which the objective function attains a "satisfactory" value and the constraints are satisfied. For example, consider the following generalization of the above model:

$$r^* := \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \ \rho(U) \tag{46}$$

$$\begin{matrix} f(x,u) \geq f^*(u) - \varepsilon \\ h(x,u) \in C \end{matrix} \quad , \ \forall u\in U \tag{47}$$

In words, we seek a decision $x^* \in X$ that yields the largest subset U of $\mathfrak{U}$ such that:

[7]See Schneller and Sphicas 1983, Eiselt et al 1998 for examples where the problem is "manageable".

- The satisficing condition $h(x^*, u) \in C$ holds for all $u \in U$.
- The ε-optimality condition $f(x^*, u) \geq f^*(u) - \varepsilon$ holds for all $u \in U$.

Clearly this model is a Maximin model. Indeed, here is the classical Maximin formulation of this model:

$$r^* := \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \min_{u\in U} \ \vartheta(x, U, u) \tag{48}$$

where

$$\vartheta(x, U, u) := \begin{cases} \rho(U) & , \ \ h(x, u) \in C \text{ and } f(x, u) \geq f^*(u) - \varepsilon \\ -\infty & , \ \ \text{otherwise} \end{cases} \tag{49}$$

for $x \in X, U \subseteq \mathfrak{U}, u \in U$.

The equivalent mathematical programming formulation is then as follows:

$$r^* := \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}\\ v\in\mathbb{R}}} \{v : v \leq \vartheta(x, U, u), \forall u \in U\} \tag{50}$$

$$= \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \{\rho(U) : \rho(U) \leq \vartheta(x, U, u), \forall u \in U\} \tag{51}$$

$$= \max_{\substack{x\in X\\ U\subseteq \mathfrak{U}}} \{\rho(U) : h(x, u) \in C, f(x, u) \geq f^*(u) - \varepsilon, \forall u \in U\} \tag{52}$$

Note that in this framework the robustness of decision $x \in X$ is prescribed as follows:

$$r(x) := \max_{U\subseteq \mathfrak{U}} \ \rho(U) \tag{53}$$

$$f(x^*, u) \geq f^*(u) - \varepsilon \ , \ \forall u \in U \tag{54}$$

$$h(x, u) \in C \ , \ \forall u \in U \tag{55}$$

Observe that this is itself is a Maximin model, namely

$$r(x) := \max_{U\subseteq \mathfrak{U}} \{\rho(U) : f(x^*, u) \geq f^*(u) - \varepsilon, \ h(x, u) \in C \ , \ \forall u \in U\} \tag{56}$$

$$= \max_{U\subseteq \mathfrak{U}} \ \min_{u\in U} \ \vartheta(x, U, u) \tag{57}$$

In the next section we consider a specialized partial robustness.

7.4 Local robustness

Consider the following much less stringent partial robust satisficing model:

$$\alpha^* := \max_{\substack{x\in X\\ \alpha\geq 0}} \ \alpha \tag{58}$$

$$h(x, u) \in C \ , \ \forall u \in U(x, \alpha) \tag{59}$$

where $U(x,\alpha)$ denotes a subset of "size" α of $\mathfrak{U}$ associated with decision x. The assumption is, of course, that $U(x,\alpha)$ is non-decreasing with α, namely for each $x \in X$ and $\alpha \geq 0$ we have

$$U(x,\alpha) \subseteq U(x,\alpha+\varepsilon) \ , \ \forall \varepsilon \geq 0 \tag{60}$$

In words, in this model the objective is to find a decision $x \in X$ such that the constraint $h(x,u) \in C$ is satisfied over the largest subset $U(x,\alpha)$ of $\mathfrak{U}$. The size

$$\alpha(x) := \max_{\alpha \geq 0} \{\alpha : h(x,u) \in C, \forall u \in U(x,\alpha)\} \ , \ x \in X \tag{61}$$

is regarded the robustness of decision x.

The classical Maximin formulation of this model is as follows:

$$\alpha(x) : = \max_{\alpha \geq 0} \min_{u \in U(x,\alpha)} f(x,u) \tag{62}$$

where

$$f(x,u) : = \begin{cases} \alpha & , \ h(x,u) \in C \\ -\infty & , \ h(x,u) \notin C \end{cases} \ , \ x \in X, u \in U(x,\alpha) \tag{63}$$

And the mathematical programming formulation is thus:

$$\alpha(x) : = \max_{\substack{\alpha \geq 0 \\ v \in \mathbb{R}}} \{v : v \leq f(x,u), \forall u \in U(x,\alpha)\} \tag{64}$$

$$= \max_{\alpha \geq 0} \{\alpha : \alpha \leq f(x,u), \forall u \in U(x,\alpha)\} \tag{65}$$

$$= \max_{\alpha \geq 0} \{\alpha : h(x,u) \in C, \forall u \in U(x,\alpha)\} \tag{66}$$

which, needless to say, is equivalent to (61).

It should be stressed that the nesting property (60) entails that the robustness in this case is local in nature. That is, the robustness of decision x is evaluated in the immediate neighborhood of the set $U(x,0)$.

For this reason, this model is unsuitable for cases where $\mathfrak{U}$ represents the region of uncertainty associated with a parameter u whose true value is subject to severe uncertainty and where $U(x,0) = \{\tilde{u}\}$ with $\tilde{u}$ denoting the value of an estimate of the true value of u.

The point is that under severe uncertainty the estimate $\tilde{u}$ is a poor indication of the true value of u and is likely to be substantially wrong (Sniedovich 2007). It is not surprising therefore that the literature on worst-case analysis and robust optimization under *severe* uncertainty *does not* bother to warn against the danger of using a *single point estimate* to represent the region of uncertainty. It would seem that this "omission" reflects the tacit understanding that such an idea is so incongruent with the basic dilemma in decision-making under severe uncertainty that it would not even be contemplated. This is why I occasionally refer to this local approach to severe uncertainty as *Voodoo decision theory*[8].

[8]See the Voodoo directory on my website (`voodoo.moshe-online.com`)

8 Maximin models in disguise

In this section I illustrate what I mean by the term "Maximin model in disguise". In a nutshell, this is a model that – for whatever reasons – does not "look" like a Maximin model, but is in fact a simple Maximin model after all.

8.1 Info-Gap's robustness model

This model (Ben-Haim 2001, 2006) is designed specifically for robust decision-making under severe uncertainty. It consists of the following constructs:

- A *parameter* $u \in \mathfrak{U}$ whose true value is unknown, except that it is an element of a given set $\mathfrak{U}$.
- An *estimate* $\tilde{u} \in \mathfrak{U}$ of the true value of u.
- A family of *regions of uncertainty* $U(\alpha, \tilde{u}) \subseteq \mathfrak{U}$ of "size" $\alpha \geq 0$ centered at $\tilde{u}$. It is assumed that the regions of uncertainty $U(\alpha, \tilde{u}), \alpha \geq 0$ are nested and are non-decreasing with α, namely

$$U(0, \tilde{u}) = \{\tilde{u}\} \tag{67}$$

$$U(\alpha, \tilde{u}) \subseteq U(\alpha + \varepsilon, \tilde{u}) \ , \ \forall \alpha, \varepsilon \geq 0 \tag{68}$$

- A *decision space* $\mathbb{Q}$ stipulating the decisions available to the decision maker.
- A *critical reward level*, $r_c \in \mathbb{R}$.
- A real-valued *performance function* R on $\mathbb{Q} \times \mathfrak{U}$.

The generic *Info-Gap* robustness model that is most relevant to our discussion is as follows:

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \ \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in U(\alpha, \tilde{u})} R(q, u) \right\} \tag{69}$$

where with no loss of generality we assume that $r_c \leq R(q, \tilde{u}), \forall q \in \mathbb{Q}$.

Note that this model seeks robustness with regard to the performance constraint $r_c \leq R(q, u)$, hence it is a robsut-satisficing model.

In the *Info-Gap* idiom,

$$\hat{\alpha}(q, r_c) := \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in U(\alpha, \tilde{u})} R(q, u) \right\} \ , \ q \in \mathbb{Q} \tag{70}$$

is the *robustness* of decision q. Observe that

$$\hat{\alpha}(r_c) = \max_{q \in \mathbb{Q}} \ \hat{\alpha}(q, r_c) \tag{71}$$

Thus, the objective is to maximize the robustness: an optimal decision is a decision possessing the greatest robustness.

In plain language then, the robustness of decision $q \in \mathbb{Q}$ is the largest value of α such that the performance constraint $r_c \leq R(q, u)$ is satisfied for all values of u in the region of uncertainty of size α, namely in $U(\alpha, \tilde{u})$.

This should ring the Maximin bell!

8.2 A Maximin perspective on Info-Gap's robustness

The Info-Gap literature presents Info-Gap theory as a new methodology, indeed as a sort of "breakthrough" in decision theory. For instance,

> Info-gap decision theory is radically different from all current theories of decision under uncertainty. The difference originates in the modelling of uncertainty as an information gap rather than as a probability. The need for info-gap modeling and management of uncertainty arises in dealing with severe lack of information and highly unstructured uncertainty.
>
> Ben-Haim (2006, p. xii)

> In this book we concentrate on the fairly new concept of information-gap uncertainty, whose differences from more classical approaches to uncertainty are real and deep. Despite the power of classical decision theories, in many areas such as engineering, economics, management, medicine and public policy, a need has arisen for a different format for decisions based on severely uncertain evidence.
>
> Ben-Haim (2006, p. 11)

What is most puzzling about these claims is that they are made without a substantiating comparative analysis between Info-Gap and classical decision theoretic methodologies such as Maximin. Indeed, none of the three Info-Gap books (Ben-Haim 1996, 2001, 2006) mention Maximin for what it is: a well established methodology for decision-making under severe uncertainty.

Elsewhere in the Info-Gap literature references are made to the Maximin paradigm. But their general drift is as follows (Ben-Haim 1999, p. 271-2):

> We note that robust reliability is emphatically not a worst-case analysis. In classical worst-case min-max analysis the designer minimizes the impact of the maximally damaging case. But an info-gap model of uncertainty is an unbounded family of nested sets: $U(\alpha, \tilde{u})$, for all $\alpha \geq 0$. Consequently, there is no worst case: any adverse occurrence is less damaging than some other more extreme event occurring at a larger value of α. What Eq. (1) expresses is the greatest level of uncertainty consistent with no-failure. When the designer chooses q to maximize $\hat{\alpha}(q, r_c)$ he is maximizing his immunity to an unbounded ambient uncertainty. The closest this comes to "min-maxing" is that the design is chosen so that "bad" events (causing reward R less than r_c) occur as "far away" as possible (beyond a maximized value of $\hat{\alpha}$).

The flaws in this thesis are discussed in Sniedovich (2007). For the purposes of this discussion suffice it to mention the following result:

Theorem 1 *Info-Gap's robustness model is an instance of Wald's Maximin model. Specifically,*

$$\max\left\{\alpha \geq 0 : r_c \leq \min_{u\in U(\alpha,\tilde{u})} R(q,u)\right\} = \max_{\alpha\geq 0} \min_{u\in U(\alpha,\tilde{u})} \alpha \cdot (r_c \preceq R(q,u)) \tag{72}$$

where the binary operator $\preceq$ is defined as follows:

$$a \preceq b := \begin{cases} 1 & , \quad a \leq b \\ 0 & , \quad a > b \end{cases} \quad , \; a, b \in \mathbb{R} \tag{73}$$

★

To confirm that this is so, note that the mathematical programming formulation of this Maximin model is as follows:

$$\alpha(q) : = \max_{\substack{\alpha \geq 0 \\ v \in \mathbb{R}}} \{v : v \leq \alpha \cdot (r_c \preceq R(q,u)), \forall u \in U(\alpha, \tilde{u})\} \tag{74}$$

$$= \max_{\alpha \geq 0} \{\alpha : \alpha \leq \alpha \cdot (r_c \preceq R(q,u)), \forall u \in U(\alpha, \tilde{u})\} \tag{75}$$

$$= \max_{\alpha \geq 0} \{\alpha : r_c \leq R(q,u), \forall u \in U(\alpha, \tilde{u})\} \tag{76}$$

$$= \max \left\{\alpha : r_c \leq \min_{u \in U(\alpha, \tilde{u})} R(q,u)\right\} \tag{77}$$

So, contrary to the claims made in the Info-Gap literature about the relationship between Info-Gap's robustness model and the Maximin paradigm, Info-Gap's robustness model is a Maximin model par excellence – in disguise.

Comments:

- Some Info-Gap users seem a bit uneasy about the above analysis, claiming that the following is also an Info-Gap robustness model:

  $$\alpha(q) := \max \left\{\alpha : r_c \geq \max_{u \in U(\alpha, \tilde{u})} R(q,u)\right\} \tag{78}$$

  So, how can this be a Maximin model given that max appears twice in the expression and there is no trace of a min here ?!?!

  In reply, here is the classical Maximin formulation of this Info-Gap model:

  $$\alpha(q) : = \max_{\alpha \geq 0} \min_{u \in U(\alpha, \tilde{u})} \alpha \cdot (r_c \succeq R(q,u)) \tag{79}$$

  where

  $$a \succeq b : = \begin{cases} 1 & , \quad a \geq b \\ 0 & , \quad a < b \end{cases} \quad , \; a, b \in \mathbb{R} \tag{80}$$

  What makes the Info-Gap model a Maximin model is the fact that the performance constraint – be it $r_c \leq R(q,u)$ or $r_c \geq R(q,u)$, or whatever – is required to be satisfied *for all* $u \in U(\alpha, \tilde{u})$. This means that the least favorable u in $U(\alpha, \tilde{u})$ determines whether α is admissible with respect to decision q. And since the decision maker is *maximizing* α, the "least favorable" u in $U(\alpha, \tilde{u})$ is one that *minimizes* $r_c \succeq R(q,u)$ with respect to u over $U(\alpha, \tilde{u})$.

- The regions of uncertainty of Info-Gap models that one typically finds in the literature are invariably "continuous" with α. For this reason, some would argue that Info-Gap's robustness model is in fact a *stability* model. This is due to the fact that it is concerned with "nice, smooth, perturbations" in the value of a parameter of the model in the neighborhood of the estimate $\tilde{u}$. Indeed, according to Jen (2003, p. 4), robustness is about perturbations that are more varied:

  > The first observation is that robustness is a measure of feature persistence for systems, or for features of systems, that are difficult to quantify, or to parametrize (i.e., to describe the dependence on quantitative variables); and with which it is therefore difficult to associate a metric or norm.

  Furthermore, the following point is particularly relevant given the *severity* of the uncertainty that Info-Gap is supposed to handle (Jen 2003, pp. 4):

  > Second, robustness is a measure of feature persistence in systems where the perturbations to be considered are not fluctuations in external inputs or internal system parameters, but instead represent changes in system composition, system topology, or in the fundamental assumptions regarding the environment in which the system operates.

  These observations notwithstanding, I call attention to the fact that in the literature of *robust optimization* the term *robustness* is commonly used to denote a region of uncertainty/variability that is nice, smooth and homogenous – typically a (convex) subset of $\mathbb{R}^n$.

  For this reason, in this discussion I use the term "local robustness" to capture some of the aspects of the distinction between *stability* and *robustness.* This terminology also explains why in the framework of decision-making under *severe* uncertainty Info-Gap decision is a *voodoo theory.*

- Note that the absolute "size" of the region of uncertainty over which a decision satisfies the performance constraint is not necessarily a good measure of the robustness of a decision. What is important is the size of this region in comparison to the size of the entire region of uncertainty. This is related to the ratio used in the robustness score formulated by Wong and Rosenhead (2000).

- Along the same lines, the distinction between local and complete robustness is analogous to the distinction between *local* and *global optimization.*

- And finally, it is appropriate to remind ourselves of Savage's (1954) *Small World - Grand World* metaphor. As indicated by Laskey and Lehner (1994, p. 1650):

  > As long as the small world model's predictions are reasonably accurate, the small world model will be a reasonable approximation to the larger world.

  The point to note in this regard is that Info-Gap's small world is an immediate neighborhood of a *wild guess.* Hence, there is no reason to believe that the results it generates are reasonable approximations to the larger world (severe uncertainty) that it is supposed to deal with.

8.3 Generalized Info-Gap robustness model

Consider next the following robustness model that is viewed as a generalization of Info-Gap's robustness model:

$$\alpha(x) := \max_{\mathcal{F}\subseteq\mathfrak{F}(x)} \{\rho(\mathcal{F}) : r \leq R(x,f), \forall f \in \mathcal{F}\} \ , \ x \in X \tag{81}$$

where

- X is some set
- $\mathfrak{F}(x)$ is a set of conditional probability density functions associated with x.
- ρ is a real-valued function on the power set of $\mathfrak{F} = \bigcup_{s\in X} \mathfrak{F}(x)$ such that

$$\mathcal{F} \subset \mathcal{F}' \rightarrow \rho(\mathcal{F}) \leq \rho(\mathcal{F}') \tag{82}$$

$$\rho(\mathcal{F}) \geq 0, \forall \mathcal{F} \subseteq \mathfrak{F} \tag{83}$$

- r is a given numeric constant.
- R is a real-valued function on $X \times \mathfrak{F}$.

In the context of Moffitt et al's (2005) generalized Info-Gap robustness model, $\rho(\mathcal{F})$ stipulates the "size" of the set $\mathcal{F}$, and $R(x,f)$ denotes the conditional expected value of some utility function basFifth-order methods for the numerical solution of ordinary differential equations. ed on the conditional probability distribution function f.

Observe that, by inspection,

$$\max_{\mathcal{F}\subseteq\mathfrak{F}(x)} \{\rho(\mathcal{F}) : r \leq R(x,f), \forall f \in \mathcal{F}\} = \max_{\mathcal{F}\subseteq\mathfrak{F}(x)} \min_{f\in\mathcal{F}} \, (r \preceq R(x,f)) \cdot \rho(\mathcal{F}) \tag{84}$$

hence this generalized Info-Gap robustness model is a Maximin model. To confirm this, note that the mathematical programming formulation of this Maximin model is as follows:

$$\begin{aligned} \alpha(x) : &= \max_{\substack{\mathcal{F}\subseteq\mathfrak{F}(x) \\ v\in\mathbb{R}}} \ \{v : v \leq (r \preceq R(x,f)) \cdot \rho(\mathcal{F}), \forall f \in \mathcal{F}\} && (85) \\ &= \max_{\mathcal{F}\subseteq\mathfrak{F}(x)} \ \{\rho(\mathcal{F}) : \rho(\mathcal{F}) \leq (r \preceq R(x,f)) \cdot \rho(\mathcal{F}), \forall f \in \mathcal{F}\} && (86) \\ &= \max_{\mathcal{F}\subseteq\mathfrak{F}(x)} \ \{\rho(\mathcal{F}) : r \leq R(x,f), \forall f \in \mathcal{F}\} && (87) \end{aligned}$$

Hence, Moffitt et al's (2005) generalized Info-Gap robustness model is equally a Maximin model in disguise.

8.4 Radius of stability

The radius of stability is a very popular local robustness measure. To the best of my knowledge, this intuitive phrase was coined by Milne and Reynolds (1962) in a paper entitled *Fifth-order methods for the numerical solution of ordinary differential equations.* The description there (page 62) is as follows:

> It is convenient to use the term "radius of stability of a formula" for the radius of the largest circle with center at the origin in the s-plane inside which the formula remains stable.

Apparently independently, the term was coined by Hinrichsen and Pritchard (1986a, 1986b) in the field of control theory. According to Paice and Wirth (1998, p.289):

> Robustness analysis has played a prominent role in the theory of linear systems. In particular the state-state approach via stability radii has received considerable attention, see [HP2], [HP3], and references therein. In this approach a perturbation structure is defined for a realization of the system, and the robustness of the system is identified with the norm of the smallest destabilizing perturbation. In recent years there has been a great deal of work done on extending these results to more general perturbation classes, see, for example, the survey paper [PD], and for recent results on stability radii with respect to real perturbations...

where HP2 = Hinrichsen and Pritchard (1990), HP3 = Hinrichsen and Pritchard (1992) and PD= Packard and Doyle (1993).

In the first edition of the *Encyclopedia of Optimization,* Zlobec (2001) describes the "radius of stability" as follows:

> The radius of the largest ball centered at θ^*, with the property that the model is stable at its every interior point θ, is the radius of stability at θ^*, e.g, [69]. It is a measure of how much the system can be uniformly strained from θ^* before it starts breaking down.

where [69] = Zlobec (1988).

In the context of our discussion of local robustness in §7.4, it is natural to define it as follows:

> The *radius of stability* of the performance requirement $h(x,u) \in C$ with respect to decision x is the largest value of α such that $h(x,u) \in C$ at $\tilde{u}, \forall u \in U(x,\alpha)$.

Not surprisingly, this is precisely the definition of $\alpha(x)$ in (61). In view of the analysis in §7.4, we conclude that the radius of stability is in fact a Maximin model in disguise.

Now consider the *Info-Gap's robustness model* formulated in (70), namely

$$\hat{\alpha}(q,r_c) := \max\left\{\alpha \geq 0 : r_c \leq \min_{u\in U(\alpha,\tilde{u})} R(q,u)\right\} \ , \ q \in \mathbb{Q} \tag{88}$$

$$= \max\left\{\alpha \geq 0 : r_c \leq R(q,u), \forall u \in U(\alpha,\tilde{u})\right\} \tag{89}$$

Clearly, $\hat{\alpha}(q,r_c)$ is the radius of stability of the requirement $r_c \geq R(q,u)$ at $\tilde{u}$ with respect to decision q.

It turns out then that *info-gap's robustness measure* is just the radius of stability of the performance requirement (Sniedovich 2010, 2012a, 2012b).

8.5 A template

The following results can serve as a gauge for determining whether an optimization model is a Maximin model.

Theorem 2

$$\max_{x\in X} \{g(x) : h(x,u) \in C, \forall u \in U(x)\} = \max_{x\in X} \min_{u\in U(x)} \xi(x,u) \tag{90}$$

$$= \max_{\substack{x\in X\\ v\in\mathbb{R}}} \{v : v \leq \xi(x,u), \forall u \in U(x)\} \tag{91}$$

where

$$\xi(x,u) := \begin{cases} g(x) & , \quad h(x,u) \in C \\ -\infty & , \quad h(x,u) \notin C \end{cases} \quad x \in X, u \in U(x) \tag{92}$$

★

This explains why Maximin is such a powerful modeling framework: the sets X and C are arbitrary and so are the sets $U(x), x \in X$.

Note that the seemingly innocent constraint $h(x,u) \in C$ can in fact represent any system of constraints.

The conclusion is therefore that the feature that in effect makes

$$\max_{x\in X} \{g(x) : h(x,u) \in C, \forall u \in U(x)\} \tag{93}$$

a Maximin model is the clause $\forall u \in U(x)$.

So here is a template for identifying a Maximin model in disguise and its two conventional Maximin counterparts:

Maximin Model in Disguise:

$$z^* := \max_{x} \ g(x) \tag{94}$$

$$s.t. \quad \text{Constraints on } x \tag{95}$$

$$\text{Constraints on } (x,u) \ , \ \forall u \in U(x) \tag{96}$$

Classical Maximin Model:

$$z^* := \max_{x\in X} \min_{u\in U(x)} f(x,u) \tag{97}$$

where

$$X := \{x : \text{Constraints on } x\} \tag{98}$$

$$f(x,u) := \begin{cases} g(x) & , \quad \text{Constraints on } (x,u) \text{ are satisfied} \\ -\infty & , \quad \text{Otherwise} \end{cases} \tag{99}$$

Maximin Model á la Mathematical Programming :

$$z^* := \max_{x,v} \ v \tag{100}$$

$$s.t. \quad v \leq f(x,u) \ , \ \forall u \in U(x) \tag{101}$$

$$\text{Constraints on } x \tag{102}$$

$$\text{Constraints on } (x,u) \ \ , \ \forall u \in U(x) \tag{103}$$

Note that, in practice, the mathematical programming formulation of the Maximin model will be simplified via the substitution of the value of $f(x,u)$ given in (99) by (101), to yield – after further simplification – the Maximin model in disguise (94)-(96).

8.6 Examples

Consider the following Info-Gap robustness model associated with a portfolio investment problem (Ben-Haim, 2006, p. 70-1) where

$$U(\alpha,\tilde{u}) := \{\tilde{u} + w : wWw \leq \alpha^2, w \in \mathbb{R}^n\} \ , \ \alpha \geq 0 \tag{104}$$

W is a square positive definite matrix of size n representing the inverse of a covariance matrix and $\tilde{u} \in \mathbb{R}^n$ is the estimate of the true value of u (future values of securities).

In the following models, r_c is a numeric constant and $q \in \mathbb{R}^n$ is the decision variable (investment vector).

Maximin model in disguise

$$\hat{\alpha}(q,r_c) := \max_{\alpha \geq 0} \left\{ \alpha : \ r_c \leq q^T u \ , \forall u \in U(\alpha,\tilde{u}) \right\} \tag{105}$$

Classical Maximin model

$$\hat{\alpha}(q,r_c) := \max_{\alpha \geq 0} \ \min_{u \in U(\alpha,\tilde{u})} \alpha \cdot \left(r_c \preceq q^T u \right) \tag{106}$$

Maximin model á la Mathematical Programming

$$\hat{\alpha}(q,r_c) := \max_{\substack{\alpha \geq 0 \\ v \in \{0,\alpha\}}} \ \left\{ v : \ v \leq \alpha \cdot \left(r_c \preceq q^T u \right), \forall u \in U(\alpha,\tilde{u}) \right\} \tag{107}$$

Next, consider the following robustness model associated with a containers inspection problem (Moffitt et al. 2005, p. 7), where N denotes the total number of containers, n denotes the number of containers inspected (decision variable), π_c denotes the critical probability of failure, and p_c denotes an upper bound on the probability that damaging material is present in one of the containers.

Maximin model in disguise

$$p(n,\pi_c,p_c) := \max_{\alpha \in [0,p_c]} \ \alpha \tag{108}$$

$$p \leq \frac{N\pi_c}{N-n} \ , \ \forall p \in [0,\alpha] \tag{109}$$

Classical Maximin model

$$p(n,\pi_c,p_c) := \max_{\alpha\in[0,p_c]} \min_{p\in[0,\alpha]} \alpha \cdot \left(p \preceq \frac{N\pi_c}{N-n}\right) \tag{110}$$

Maximin model á la Mathematical Programming

$$p(n,\pi_c,p_c) := \max_{\substack{\alpha\in[0,p_c]\\ v\in\{0,\alpha\}}} v \tag{111}$$

$$v \leq \alpha \cdot \left(p \preceq \frac{N\pi_c}{N-n}\right), \forall p \in [0,\alpha] \tag{112}$$

Note that, as observed by Moffitt et al. (2005, p. 7), the optimization problem under consideration is trivial and can be solved by inspection, yielding

$$p(n,\pi_c,p_c) = \frac{Np_c}{N-n} \tag{113}$$

Comment:
I note in passing that the trivial nature of the solution to the robustness problem is not a rare event. To see why this is so, consider the case where the regions of uncertainty $U(\alpha,\tilde{u}), \alpha \geq 0$ are intervals of the real line. Then very often the problem posed by Info-Gap's generic robustness model

$$\hat{\alpha}(q,r_c) := \max_{\alpha\geq 0} \{\alpha : \ r_c \leq R(q,u) \ , \forall u \in U(\alpha,\tilde{u})\} \tag{114}$$

is very easy and can be solved by inspection.

In particular, this will be the case if $R(q,u)$ is continuous and strictly increasing with u. Under these conditions $\hat{\alpha}(q,r_c)$ can be determined by solving $R(q,u) = r_c$ for u and then using this critical value of u to determine the largest value of α such that the critical value of u is the upper end-point of the interval $U(\alpha,\tilde{u})$. This is a recurring theme in Info-Gap robustness analyses.

Furthermore, in such cases the critical value of u will be invariant with the value of the estimate $\tilde{u}$, assuming as usual that $r_c \leq R(q,\tilde{u})$. In fact, the critical value of u is independent of the structure of the uncertainty regions $U(\alpha,\tilde{u})$. This means that the robustness problem can be stated as follows:

$$\text{Find a } u \in \mathfrak{U} \text{ such that } R(q,u) = r_c \tag{115}$$

In such cases the use of Info-Gap's robustness model is counter-productive as it obscures the trivial nature of the problem under consideration. Indeed, the critical value of u can be regarded as the robustness of decision q and there is no need to introduce the somewhat artificial regions of uncertainty.

8.7 Model vs Problem

It is important to remember that the selfsame problem often lends itself to more than one formulation (model). Furthermore, that these various formulations (models) can be quite different from one another. For instance, the linear programming and dynamic programming formulations of the classical traveling salesman problem are a case in point.

Pursuing this line of argument, the idea here is that with the aid of *dummy variables* any "conventional" optimization model can be formulated as a Maximin model:

$$\max_{x\in X} f(x) = \max_{x\in X} \min_{u\in U} g(x,u) \tag{116}$$

where for any $x \in X$

$$g(x,u) = f(x) \ , \ \forall u \in U \tag{117}$$

The implication is therefore that a problem that is amenable to a Maximin formulation may have a number of other formulations that are quite different from its formulation as a Maximin model.

I note this fact to emphasize that the discussion in the preceding sections deals with models not problems.

Thus, the "Model vs Problem" issue adds a new dimension to the formulation of Maximin models. It is plausible that a problem amenable to a Maximin formulation would have an alternative, indeed more attractive, formulation but this more attractive model would not easily be derived from the Maximin formulation of the problem.

One need hardly point out that this issue is not specific to the formulation of Maximin models. It is an issue that is commonly encountered in mathematical modeling as such.

Hence, it would be wise to remember the First Commandment of mathematical modeling:

Thou shalt not fall in love with thy model!

That is, one of the most important aspects of the art of mathematical modeling is to decide which model is suitable most for the problem under consideration. Hence, the question whether to use or not to use a Maximin model to formulate a given problem is a fundamental modeling issue.

8.8 Explicit handling of constraints by the classic Format

It should be pointed out that the use of a "penality" function (92) in Theorem 2 does not mean that the classic format of the Maximin model cannot handle constraints *explicitly.* The point is that constraints such as $h(x,u) \in C, \forall u \in U(x)$ can be incorporated explicitly (as "constraints") in this format, in which case the format retains the original objective function $g = g(x)$. In other words, such constraints can be incorporated explicitly in the definition of the decision space of the Maximin model without resorting to a penalty function.

Theorem 3

$$\max_{x\in X} \{g(x) : h(x,u) \in C, \forall u \in U(x)\} = \max_{x\in \widehat{X}} \min_{u\in U(x)} g(x) \tag{118}$$

where

$$\widehat{X} := \{x \in X : h(x,u) \in C, \forall u \in U(x)\} \tag{119}$$

★

By definition, $\widehat{X}$ is the set of feasible decisions associated with the robustness (worst case) constraint $h(x,u) \in C, \forall u \in U(x)$.

I mention this obvious modeling aspect of the classic Maximin format for the benefit of novices who may regard the use of a penalty function such as (92) as somewhat idiosyncratic (e.g. Davidovitch 2009, p. 18).

9 Solution methods and algorithms

For obvious reasons, no general purpose solution methods designed to solve Maximin problems are available. Some Maximin problems are easily solved, some are extremely difficult to solve.

This fact highlights the importance that one should place on "simplifying" the mathematical programming formulation of a Maximin model. The goal of this "simplification" is to obtain a "standard" optimization model that would then render the problem in question amenable to any one of the relevant existing solution methods.

For our purposes it suffices to note that the choice of state spaces (regions of uncertainty) for a Maximin model may have far reaching consequences insofar as solution methods and algorithms are concerned.

Details regarding the algorithmic and computational aspects of Maximin models can be found in the robust optimization literature and elsewhere (e.g. Ben-Tal et al 2006, 2009; Demyanov and Malozemov 1990; Du and Pardalos 1995; Kouvelis and Yu 1997; Rustem and Howe 2002; Vladimirou and Zenios 1997; Zhukosky Salukvadze 1994).

Some of the software tools provided by DECISIONARIUM[9] are Maximin based. And the extensive work at the RAND corporation on decision support for robust decision-making (e.g. Lempert et al 2003, 2006) is also relevant to this discussion.

10 Conclusions

Despite the austere simplicity of its formulation, the generic Maximin model puts at our disposal a highly flexible modeling tool. The framework it provides for modeling decision-making under severe uncertainty proves particularly appropriate for the treatment of robust satisficing, robust optimizing, and robust optimizing and satisficing problems.

As we have seen, the Maximin paradigm gives significant leeway to enable control of the degree of robustness sought. Therefore, the Maximin paradigm as such need not necessarily be overly conservative.

[9]See www.decisionarium.net

Info-Gap users should take note that, contrary to persistent statements made in the literature, Info-Gap's generic robustness model is a Maximin model. However, given that its definition of robustness is in principle local, as a methodology Info-Gap decision theory is thoroughly unsuitable for robust decision-making under severe uncertainty (Sniedovich 2007, 2010).

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