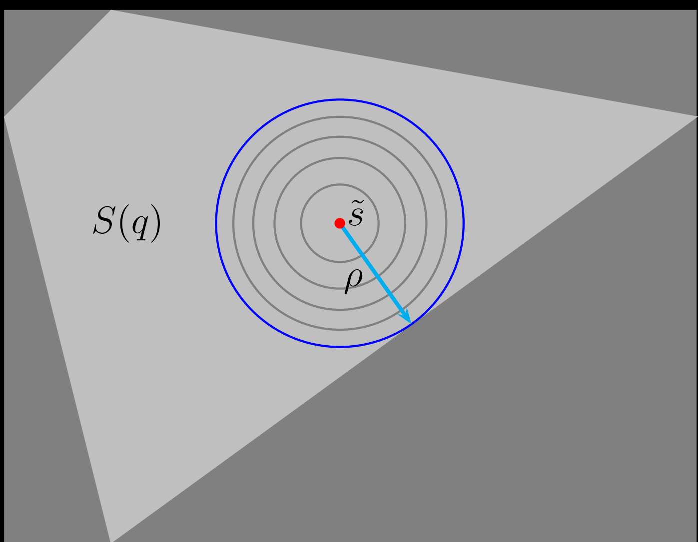
Info-gap Decision Theory

A Perspective from the Land of the Black Swan

a comprehensive critique of info-gap decision theory

Phascolarctos cinereus

Moshe Sniedovich



$$\max\ \{\rho \geq 0 : s \in S(q), \forall s \in B(\rho, \tilde{s})\}$$

Info-Gap Decision Theory:
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Foreword

It won't take readers of this document very long to appreciate its context as part of Moshe Sniedovich's campaign to alert the research community to the dangers of using Information-Gap (Info-Gap) Decision Theory as a tool to aid decision-making in situations of severe uncertainty. Moshe's campaign, and the response to it from some parts of the 'Info-Gap community', have involved a great deal of passion from the protagonists, much more than is usual in the mathematical sciences.

Info-Gap Decision Theory is designed to be used in circumstances where the payoff $r(q, u)$ generated by a decision q depends on the value of an unknown parameter u. Although the decision-maker has available an estimate $\tilde{u}$, u itself is subject to 'severe uncertainty' within some space $\mathcal{U}$. The decision-maker has a performance requirement r^*, and makes the decision q that maximises the 'radius' α of the neighborhood $U(\alpha, \tilde{u})$ around the estimate $\tilde{u}$ for which $r(q, u) \geq r^*$ for all $u \in U(\alpha, \tilde{u})$.

Moshe has a number of criticisms of Info-Gap Decision Theory, which you will learn about when you read the document. As a stochastic modeller by trade, my problem with Info-Gap Decision Theory is its claim to be non-probabilistic. Indeed, Info-Gap Decision Theory makes no-attempt to describe the uncertainty in the parameter u by modelling it with a probability measure on the space $\mathcal{U}$. It is just taken for granted that it is a good idea to make a decision that maximises the radius α described above.

Of course, we all make decisions under conditions of uncertainty every day, and it clearly would be silly for me to suggest that every such decision should be supported by a stochastic model, with explicit modelling of the distributional characteristics of the uncertainty. However, if one is going to use a mathematical model, then its assumptions should be clearly-stated and logical deductions made within the context of the model should follow from the assumptions. I would claim that, without a probabilistic description of uncertainty, the logical justification for the choice of the decision q as described above is absent.

It is entirely possible that, by making suitable assumptions about the distribution of the parameter u, the operational method used in Info-Gap Decision Theory can be given a justification in terms of maximising the probability that $r(q, u) \geq r^*$. Personally, I think that such an approach to decision-making – maximising the probability that a decision is acceptable – is reasonable in many situations. However the assumptions justifying the use of method should be clear for all to see, and the decision-maker should be able to decide whether these assumptions are satisfied in his/her particular real-world situation.

In his preface, Moshe suggests that many readers will find the style of his criticisms harsh. It is certainly true that much of the text that follows is combative. However, it is not boring, and it is important for potential users of Info-Gap Decision Theory to consider the issues that are raised by Moshe. I recommend that you do not let the style of the text get in the way of the message that it carries.

Peter Taylor
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November 25, 2011

Preface

On numerous occasions over the past eight years I was asked to comment on various aspects of info-gap decision theory. Yet, despite my damning critic of this theory, it was embraced by environmental risk analysts in Australia as a practical tool for the management of severe uncertainty.

In response, at the end of 2006 I launched a campaign to contain the spread of info-gap decision theory in Australia. But, at the same time, I continued my discussion with risk analysts in Australia (and overseas) on the technical and conceptual aspects of info-gap decision theory.

This document, which gives a highly critical assessment of info-gap decision theory, was inspired to a large extent by this lengthy discussion. Hence, the analysis presented in this document is intended primarily for the benefit of info-gap scholars affiliated with the Australian Centre of Excellence for Risk Analysis (ACERA). Still, it should also be of interest to users of info-gap decision theory and to analysts contemplating using it.

The analysis in this document addresses the following three questions:

- Is info-gap decision theory the theory that it is claimed to be and does it indeed do what it is claimed to do?
- What is the role and place of info-gap decision theory in decision theory and robust decision-making in the face of severe uncertainty?
- What are the implications of the answers to these questions?

My overall conclusion is that info-gap decision theory is very problematic, and in view of this I recommend ways to deal with this issue. That is, much as this document is a critique, and a critique, by its very nature, can have the connotation of being censorious, in this case the critique is constructive and edifying. Thus, one of the main objectives of the document is to focus on the important lessons that can be drawn from the info-gap experience.

It is also important to note that the need for such a critique is very real indeed, as it is vital to counterbalance the many uncritical reports, articles, workshops, etc. on info-gap decision theory, with a comprehensive, rigorous, critical assessment of this theory. In fact, such a critique is long overdue (see Appendix I).

My original plan was to write this document in cooperation with ACERA. However, as this did not work out, I decided to go it alone. I plan to document my "info-gap experience", as I call it, in a book, which is currently under preparation. An early draft of this document served as the initial draft of the book.

The critique outlined in this document draws on the material discussed in Sniedovich (2007, 2008, 2008a, 2008b, 2009, 2009a, 2010, 2011), on material posted on my website[1], and on numerous discussions that I had over the last eight years with info-gap scholars in Australia and overseas, many of whom are affiliated with ACERA.

To enable a proper assessment of this critique, I want to point out the following.

In a recently published article in *Forest Policy and Economics*, Hildebrandt and Knoke (2011, p. 12) make the following comment about my critique of info-gap decision theory:

> However, the critique by Sniedovich (2010), although often presented overly harsh, should be taken seriously.

I am well aware that many info-gap scholars hold that my criticism of info-gap decision theory is too harsh. I take it that this charge applies less to the content of the arguments that I adduce to refute it, than to the style in which these arguments are formulated. It applies mostly to the unforgiving language that I use to bring out the real facts about this theory.

My response to this is as follows:

- My criticism is indeed harsh, but its harshness is fully justified.
- My criticism should be taken very seriously, rather than just seriously.

I want to explain briefly why my criticism of info-gap decision theory strikes its adherents as being “overly harsh”.

Very broadly, my justification for the harshness of my criticism of info-gap decision theory is simply that this is a reflection of two facts:

- The seriousness of the flaws in the theory.
- The unsubstantiated, often erroneous, statements made by info-gap scholars about this theory and related theories.

In fact, I need do no more than quote whole passages from the info-gap literature itself to bring out how flawed this theory is, and to show that erroneous claims about it are circulated uncritically in the info-gap literature. So, be ready for the large number of quotes from the info-gap literature in this document.

The point is, though, that because my refutation of this theory uses arguments that expose how unfounded, invalid, self-contradictory, etc. are some of the main claims made by info-gap scholars and the explanations they give in support of the theory, that my criticism appears to them as harsh. As you will see, though, harsh or not, my criticism is always backed up by airtight technical arguments.

So I suggest that readers of this document focus on the technical merit of my criticism and its validity, even if they disagree with my style. They will notice that my criticism is always to the point, rigorous, and fair.

I welcome comments, especially constructive criticism, on any aspect of this document. Such feedback should be of benefit to my work on the book that I am currently writing on this topic.[2]

[1] See http://info-gap.moshe-online.com

[2] For the record, I should point out that at the end of August 2011, I sent an advanced draft of this document to members of ACERA’s Advisory Scientific Committee for comments. As I received no comments on this draft, the final document is very similar to the advanced draft.

I would like to acknowledge with appreciation the support for this project that I received from the Department of Mathematics and Statistics. I particularly thank Professor Peter Taylor for his support and advice, especially after my retirement at the end of 2007, for his comments on various drafts of this document, and for the *Foreword.*

I also acknowledge the contributions of my former students, especially Daphne Do and Jaeger Renn-Jones, to my work on this subject.

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Chapter 1

Introduction

Info-gap decision theory appeared on the Australian scene in 2003 and was quickly taken up by Australian scholars in the fields of applied ecology and conservation biology, to serve as the prime tool of robust decision-making in the face of severe uncertainty in these areas. This is evidenced in the raft of activities (workshops, lectures, seminars), and publications (research reports, refereed articles, theses, research/project grants), associated with this theory (see Appendix K).

For an idea of the high regard accorded to info-gap decision theory by applied ecology and conservation biology scholars in Australia, consider this statement (emphasis is added):

> In summary, we recommend info-gap uncertainty analysis as a **standard practice** in computational reserve planning. The need for robust reserve plans may change the way biological data are interpreted. It also may change the way reserve selection results are evaluated, interpreted and communicated. Information-gap decision theory provides a **standardized methodological framework** in which implementing reserve selection uncertainty analyses is relatively straightforward.
>
> Moilanen, A., Runge, M.C., Elith, J., Tyre, A., Carmel, Y., Fegraus, E., Wintle, B., Burgman, M., and Ben-Haim, Y. (2006, p. 123)

Note that four out of the paper's nine co-authors, including Prof. Yakov Ben-Haim — the Father of info-gap decision theory — are *AEDA*[1] *Core Researchers* and are affiliated with ACERA.

Next, consider the following quote from the publisher's flyer[2], announcing the publication of the second edition of the book on *info-gap decision theory* (Ben-Haim, 2006):

> ”Of fundamental importance to all applied sciences.”
>
> Prof. Mark Burgman, School of Botany, University of Melbourne.

And consider this quote regarding the article *Robust decision-making under severe uncertainty for conservation management* by Regan et al. (2005):

[1] Applied Environmental Decision Analysis, www.aeda.edu.au

[2] See http://www.technion.ac.il/yakov/flyer02final.pdf (downloaded on March 17, 2011)

> This is the first application of a relatively new method for making decisions under uncertainty, information gap ('info-gap') theory, to a real conservation problem, saving the Sumatran rhino Dicerorhinus sumatrensis. It provides hope for managers who want to use rational decision-making methods, but are overwhelmed by the amount and types of uncertainty they face. Using 'info-gap' theory, the best decision is the one that achieves an acceptable outcome with the greatest level of uncertainty. The application is concise and ideally suited to teaching and technology transfer.
>
> Hugh Possingham: Faculty of 1000 Biology, 22 Feb 2006
> http://www.f1000biology.com/article/id/1031061/evaluation

The quote is taken from the website of *Faculty of 1000*. The following is the first paragraph of the *About* page on the *Faculty of 1000* website:

> **What is F1000?**
>
> POST-PUBLICATION PEER REVIEW
>
> The core service of Faculty of 1000 (F1000) identifies and evaluates the most important articles in biology and medical research publications. The selection process comprises a peer-nominated global 'Faculty' of the world's leading scientists and clinicians who rate the best of the articles they read and explain their importance.
>
> http://f1000.com/about/whatis
> Downloaded March 5, 2011

Not surprisingly, Regan et al. (2005) is treated in info-gap publications as a seminal paper dealing with problems of applied ecology and conservation biology. For instance (emphasis added),

> Using this approach to the problem of species conservation, we would evaluate each option — translocation, new reserve, captive breeding etc — in terms of the amount of uncertainty which our models permits for achieving an acceptable result (Regan et al. 2005). **This is a remarkable departure** from an expected utility model where any decision is based on averaging the utility of fortunate and disastrous, likely and improbable scenarios. The utility model is firmly entrenched in economics and decision theory, but it's inappropriate for many environmental management problems.
>
> Sprenger (2011, p. 9)

I need hardly point out that great importance is attributed to peer-reviewed articles (such as Regan et al. (2005)) in Australia (and elsewhere), as attesting to the merit of the research. Thus, consider the following passage in *Decision Point*, a monthly magazine of the *The Applied Environmental Decision Analysis* (AEDA) hub, that publishes articles, views and ideas on environmental decision making, biodiversity, conservation planning and monitoring (emphasis added):

> The Federal government took a calculated risk investing in a multidisciplinary centre that was very different from traditional ecological science.

> And what has been the return on that investment? Quite a lot if you consider our achievements (many of which have been presented in *Decision Point,* see the next page for just a few examples). It's important to note that all of these outputs appeared in the **peer-reviewed literature** (including some of the top journals like *Science* and *Nature*). We often forget that the CERF program is a research program, albeit applied research, and **research must eventually be subject to peer review to be credible.**
>
> Possingham (2010, p. 2)

So, one of the important goals of this document is to reflect on the credibility of peer-reviewed info-gap publications in the areas of interest to applied ecologists and conservation biologists in Australia.

For the record, I should stress though that the enthusiastic, uncritical assessment of info-gap decision theory has not been limited to Australian scholars. Thus, consider these quotes from the same publisher's flyer:

> "A landmark book . . . a new and revolutionary approach for tough decision problems when little information is available".
>
> Prof. Keith Hipel, Dept. of Systems Design Engineering, University of Waterloo.
>
> "Revolutionary strategy implications."
>
> Prof. Clifford C. Dacso, MD, MBA, Distinguished Research Professor, University of Houston and Methodist Hospital Research Inst.

Now, parallel to its growing popularity in Australia, especially in the fields of applied ecology and conservation biology, I have been unrelenting in my vociferous criticism of the theory.

At the beginning of 2006 I realized that, despite my "behind the scenes" constructive criticism, the promotion of the theory in Australia continued unabated. So at the end of 2006 I launched a campaign to contain the spread of info-gap decision theory in Australia[3]. And, I published a number of articles spelling out the fundamental flaws in this theory (Sniedovich 2006, Sniedovich 2007, 2008, 2008a, 2008b, 2009, 2009a, 2010, 2011).

I should also point out that in August 2007 I gave a seminar entitled *What exactly is wrong with Info-Gap? A Decision Theoretic Perspective,* at the Department of Mathematics and Statistics, The University of Melbourne, and a presentation entitled *A Critique of Info-Gap,* at the SRA 2007 Conference in Hobart. In both lectures I explained in detail the major flaws in *info-gap decision theory*. Prof. Yakov Ben-Haim attended both lectures. In May 4, 2007 I gave a seminar entitled *The Art and Science of Decision-Making Under Severe Uncertainty* at ACERA where I discussed, among other things, the technical and conceptual flaws in info-gap decision theory.

A year later, in August 2008, I issued "A Call for the Reassessment of the Use and Promotion of Info-Gap Decision Theory" (Sniedovich 2008c).

But despite the lectures, the articles, the "coffee sessions", and e-mail exchanges with info-gap scholars on this topic, where I carefully explained the obvious flaws in the theory, the same

[3] See `info-gap.moshe-online.com`

unsubstantiated, erroneous statements continue to circulate in the info-gap literature. I address this issue in Chapter 7.

It should be pointed out that Prof. Ben-Haim has had many opportunities over the past five years to address the criticism directed at his theory. The compilation of FAQs about info-gap decision theory (Ben-Haim, 2007) seems to be such an attempt. The analysis presented in this document takes into account Ben-Haim's documented position vis-a-vis this criticism (see Appendix G).

The objective of this document is then to contrast the following two assessments of info-gap decision theory:

1. Info-gap decision theory offers a new, reliable method for robust decision-making under severe uncertainty in areas ranging from applied ecology, conservation biology, environmental management, bio-security, homeland security to finance and economics.
2. Info-gap decision theory is a re-invention of an old, well-established paradigm, that in effect amounts to a misapplication of this famous paradigm.

My overall conclusion is that info-gap decision theory, as portrayed and formulated in Ben-Haim (2001, 2006, 2010), and as its application is described in numerous other publications, is seriously flawed both methodologically and technically. I clarify what the problematic issues are and I recommend ways for dealing with them.

This raises the obvious question:

> Since info-gap decision theory proves utterly unsuitable for dealing with the problem of robust decision-making in the face of severe uncertainty of the type that it stipulates, what theories/methods do offer a **sound** treatment of this task?

Although this question does not fall within the scope of this document, I do recommend an obvious starting point. Namely, I call the reader's attention to the vast literature on *robust decision-making under uncertainty,* particularly to the *robust optimization* literature[4].

This document was written first and foremost for Australian scholars and analysts who are already familiar with info-gap decision theory and are keen to get a clear picture of the issues that render this theory problematic.

I recommend that it be read in sequence, with short excursions to the relevant appendices, if necessary. Some readers may prefer to go directly to Chapter 7, where this discussion is summarized.

I deliberately sought to make the discussion as math-free as possible. However, it must be appreciated that info-gap decision theory is based on a mathematical model of robustness. Therefore, any rigorous examination of the theory requires some formal mathematical analysis.

Finally, although readers who are familiar with info-gap decision theory may find it surprising that my description and assessment of this theory are in stark contradiction to those put forward in the info-gap literature, it is important to keep in mind that all the statements, claims and judgments on info-gap decision theory in this document are backed up by a careful, rigorous, formal analysis of the theory.

[4]Note that although info-gap's robustness model is a robust optimization model, it proves unsuitable for the treatment of severe uncertainty of the type that it claims to take on, hence this document.

Additional resources

- ACERA's website: http://www.acera.unimelb.edu.au
- AEDA's website: http://www.aeda.edu.au
- Ben-Haim's website: http://www.info-gap.com
- Sniedovich's website: http://info-gap.moshe-online.com
- Discussion of the CSIRO Report: http://info-gap.moshe-online.com/csiro.html
- Web-page dedicated to this document: http://info-gap.moshe-online.com/acera.html

1.1 The main issues

In Ben-Haim (2001, 2006, 2010) and in other publications, info-gap decision theory is described as a new decision theory designed to seek decisions that are robust against severe uncertainty. Furthermore, this theory is claimed to be radically different from existing "conventional" theories (emphasis is added):

> Info-gap decision theory is **radically different** from **all** current theories of decision under uncertainty. The **difference** originates in the modelling of uncertainty as an information gap rather than as a probability. The need for info-gap modeling and management of uncertainty arises in dealing with severe lack of information and highly unstructured uncertainty.
>
> Ben-Haim (2006, p. xii)

> In this book we concentrate on the **fairly new** concept of information-gap uncertainty, whose **differences** from more classical approaches to uncertainty are **real and deep.** Despite the power of classical decision theories, in many areas such as engineering, economics, management, medicine and public policy, a need has arisen for a **different** format for decisions based on severely uncertain evidence.
>
> Ben-Haim (2006, p. 11)

And yet, nowhere does Ben-Haim (2001, 2006) specify what the "rival" decision theories are, nor does he make it clear in what way is info-gap decision theory radically different from current mainstream theories for robust decision-making under severe uncertainty. In fact, strange though it may sound, although the problem addressed by info-gap decision theory is a simple *robust optimization problem,* there is not a single reference in Ben-Haim (2001, 2006, 2010) to the field of *robust optimization*.

But more than this, the treatment of severe uncertainty of the type postulated by info-gap decision theory requires a *global* assessment of robustness[5]. Yet, a superficial examination suffices to reveal that the model deployed by info-gap decision theory to assess the robustness of decisions against uncertainty, by definition, seeks *local* — rather than *global* — robustness.

So, what all this adds up to is that a proper evaluation of info-gap decision theory calls for a careful examination of at least these three questions:

[5] As in local vs global optimization, local vs global economy, local vs global anesthetic, local vs global weather, local vs global news, etc.

- Does info-gap decision theory do what it claims it does?
- What is the role and place of info-gap decision theory in decision theory and robust decision-making in the face of severe uncertainty?
- What are the implications of the answers to these questions?

I address these issues in subsequent chapters of this document. But first, I consider a simple illustrative example, intended primarily to clarify the meaning of the term "severe uncertainty".

1.2 Illustrative example

What I want to highlight through this stylized example is the fundamental questions that enter the analysis of decision-making situations that are subject to *severe uncertainty*. To do this effectively, it is important to avoid becoming embroiled in technical considerations that are not vital for the clarification of these central questions. To this end, I consider a simple example that will also enable illustrating the modeling of the seemingly intuitive notion *robustness*. I shall refer to this problem as the *Threat Problem*.

> **Problem description:**
> A decision needs to be made as to what plan to adopt in the face of an anticipated bio/homeland security threat in a large metropolitan area, call it Met. There are six alternative plans, denoted A, B, C, D, E, and F.
>
> The exact location of the threat is subject to *severe uncertainty.* Therefore, the plan sought is that which assures *robustness* against this severe uncertainty.
>
> The task is then to rank the six plans according to the robustness that they provide against the severe uncertainty in the exact location of the bio/homeland security threat.
>
> Figure 1.1 shows the "safe" regions associated with the plans, where a "safe" region of a plan is that region in the metropolitan area which consists of the locations where the plan can successfully meet the threat, should it take place there. The large squares represent the metropolitan area, and the shaded regions represent the "safe" regions of the respective plans. As indicated by Plan D, the "safe" regions are not assumed to have "regular" shapes.

As my objective here is to clarify the fundamental issues facing decision makers in such situations, I shall not rush to rank the plans according to the peculiar characteristics of a plan's "safe" region.

Rather, the first question I pose is this:

> How do we approach the *modeling* of a situation such as the one described by the above problem?

I shall address this question in stages. But before I can begin, we must be clear on the following:

> What is the meaning of "robustness against severe uncertainty"? How do we define this intuitive notion quantitatively?

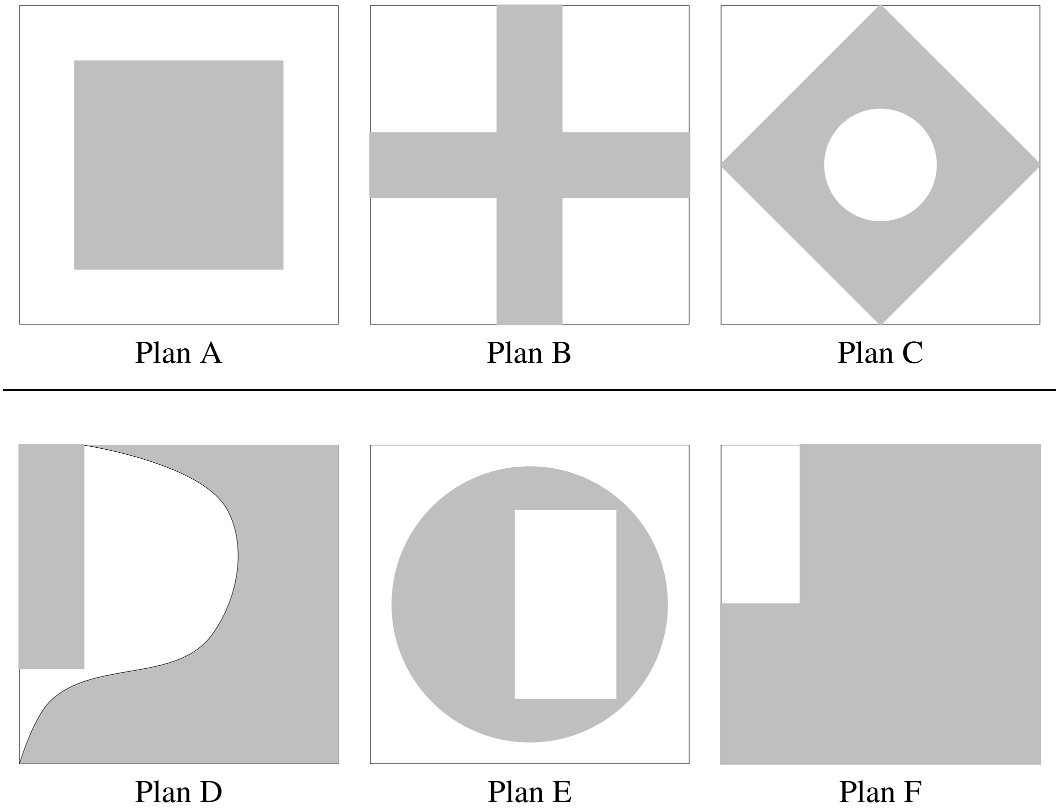


Figure 1.1: Safe regions (shaded areas) of the six plans

In particular, how robust are the six plans against the severe uncertainty in the exact location of the threat? Which one is the most robust?

Chapter 2

Robustness

It is important to appreciate that although in ordinary usage, the terms "robust" and "robust against severe uncertainty" may well be self-evident, this may not be the case when they are used to designate *mathematical/technical properties.* This means that it is imperative to make their meaning, in the context of a given mathematical model, crystal clear. Specifically, it is imperative to make it clear whether these terms are used to designate *local* robustness or *global* robustness, because a fundamental difference exists between these two types of robustness.

Now, in this document I discuss the notion of *local* robustness in considerable detail even though models of local robustness are unsuitable for the treatment of severe uncertainty of the type stipulated by info-gap decision theory. The reason that I do this is due to the fact that the definition of robustness employed by info-gap decision theory is that of *local* robustness.

To illustrate the difference between *local* and *global* robustness, consider the following simple problem:

> **Treasure Hunt problem**:
> An ancient treasure is rumored to be hidden somewhere on a certain island. The location of the treasure on the island is unknown, namely it is subject to uncertainty. A number of plans to search for the treasure are put forward. The task is to determine the robustness of each plan against the uncertainty in the (unknown) location of the treasure.

Let us consider now three distinct cases, shown in Figure 2.1. Each case exemplifies our knowledge regarding the location of the treasure and the type of robustness analysis that is required, accordingly.

As we shall see, info-gap decision theory draws no distinction between *local* and *global* robustness. Furthermore, it conducts a *local* robustness analysis in cases where a *global* analysis is required.

That said, it is important to point out that the **property** "robustness" does not obtain its meaning from an affiliation with **uncertainty.** That is, uncertainty is not the only factor against which robustness is sought. Designating as it does resilience to changes and/or perturbations[1], the property *robustness* can be defined and applied in situations that have got nothing to do with uncertainty, let alone *severe* uncertainty.

[1] See discussion in Appendix B.

Case 1: Certainty
The exact location of the treasure is known. This case is of no interest to us in this discussion. All the same, I take note of it to make the point that even if the uncertainty is completely eliminated, the decision problem can still prove difficult to solve.

Case 2: The uncertainty is very mild
Suppose that we have a pretty good *point estimate* of the exact location of the treasure. In this case it may be appropriate to conduct the robustness analysis in the *neighborhood* of the estimate. Of course, the choice of the size and shape of this neighborhood may depend on how good the "pretty good" estimate actually is. This would be a *local* robustness analysis.

Case 3: The uncertainty is severe
Suppose that we have no information on the location of the treasure, except that it is somewhere in the island. Then it may be appropriate, perhaps even necessary, to conduct the robustness analysis over the entire island. This would be a *global* robustness analysis.

Figure 2.1: Three cases of the Treasure Hunt Problem

In fact, it is far more edifying to explain the definition of robustness employed by info-gap decision theory in a framework that does not refer to uncertainty at all. Also, it is more illuminating, at this stage, to talk about "stability" rather than "robustness".

So, as a general framework for a formal definition of robustness, consider a system $q \in Q$ whose *state* $s \in S(q)$ can be either *stable* or *unstable*, and let

- Q = set of systems under consideration.
- $S(q)$ = set of all the possible/plausible states associated with system q.
- $S_{stable}(q)$ = subset of $S(q)$ consisting of all the *stable* states in $S(q)$.
- $S_{unstable}(q)$ = subset of $S(q)$ consisting of all the unstable states in $S(q)$.
- $s^* \in S_{stable}(q)$ = *nominal state.*

Formally then, $S(q)$ is the union of the two disjoint sets $S_{stable}(q)$ and $S_{unstable}(q)$. We shall refer to $S_{stable}(q)$ as the *region of stability* of system q, and to $S_{unstable}(q)$ as the *region of instability* of system q.

This is shown schematically in Figure 2.2, where the rectangle represents the state space, $S(q)$, of system q and the shaded area represents the set of stable states, $S_{stable}(q)$, namely the region of stability system q.

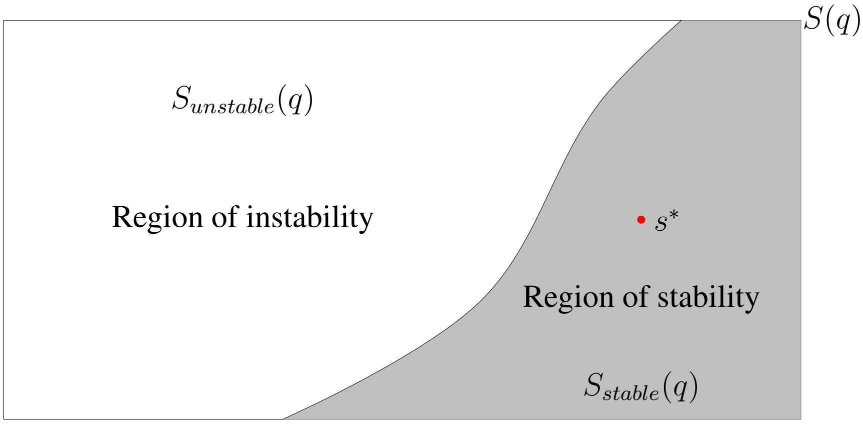


Figure 2.2: Region of stability of system q

To explain the difference between *local* and *global* stability in this context, assume that we seek a quantitative measure to gauge the system's stability. We can then ask the following two related — but distinctly different — questions:

Q1: How stable is system q against *perturbations* in the value of the *nominal state* s^*?

Q2: How stable is system q *over its state space,* $S(q)$?

The difference between these two questions is that the first asks a typical *local stability* question. That is, here the objective is to determine the stability of the system in the *locale* of a given state, namely in the immediate neighborhood of the nominal state s^*.

By implication, Q1 is not concerned (explicitly) with the stability of the system in neighborhoods of $S(q)$ that are "distant" from s^*.

In contrast, Q2 is not concerned with the stability of the system in any one particular neighborhood of $S(q)$: it is concerned with the stability of the system over the *entire state space* of system q, namely $S(q)$. This is a *global* stability question.

The distinction between local and global stability (robustness) is reminiscent of the well known distinction between *local* and *global* **optimum**.

2.1 Radius of stability

Immediately relevant to this discussion is the *Radius of Stability* model (circa 1960) — by far the most widely used model of *local* stability/robustness. It is a staple in fields such as numerical analysis, applied mathematics, control theory, economics, and parametric optimization (See Sniedovich 2010, 2011 and the references therein). The following example illustrates this intuitive concept.

2.1.1 Example

A fundamental problem in *geometry* is to find the radius of the largest circle, centered at a given point, that is contained in a polygon $P \subset \mathbb{R}^2$, where $\mathbb{R}$ denotes the real line.

Figure 2.3 illustrates this fundamental concept. It shows a polygon P, represented by the shaded area, and four circles. Note that each circle is the largest circle, centered at the specified point, that is contained in P. The radii of these circles are the *Radii of Stability* of the four points in P. Clearly, the *Radius of Stability* is a *local* property of the polygon: it is contingent on the location of the given *center point* in the polygon. Note that none of the circles shown is the largest (possible) circle contained in P.

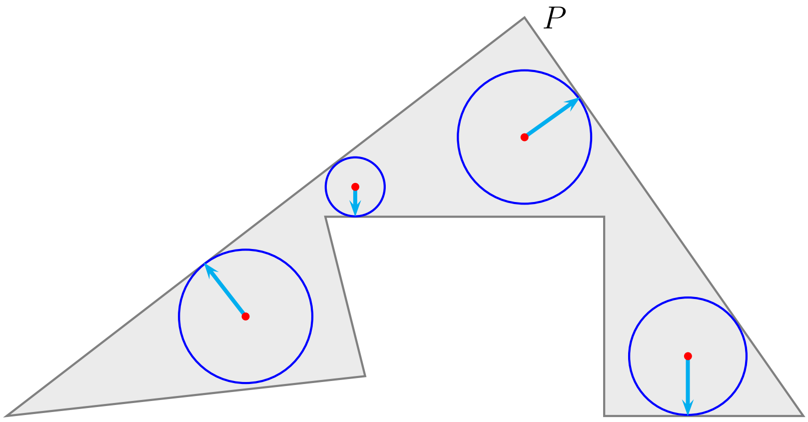


Figure 2.3: Radii of Stability of four points in a polygon

Conceptually then, the concept *Radius of Stability* addresses the following question:

> Given a point $\tilde{p} \in P$, what is the radius of the largest circle centered at $\tilde{p}$ that is contained in P?

Let $\rho(\tilde{p})$ denote this largest radius: this is the *Radius of Stability* of P at $\tilde{p}$. More formally,

$$\rho(\tilde{p}) := \max \{\rho \geq 0 : C(\rho, \tilde{p}) \subseteq P\} \tag{2.1}$$

$$= \max \{\rho \geq 0 : p \in P, \forall p \in C(\rho, \tilde{p})\} \tag{2.2}$$

where $C(\rho, \tilde{p})$ denotes a circle of radius ρ centered at $\tilde{p}$.

The following is then an informal definition of the *Radius of Stability* in this simple case:

> The *Radius of Stability* of P at $\tilde{p} \in P$ is the radius of the largest circle centered at $\tilde{p}$ that is contained in P.

Also note that the *Radius of Stability* of P at a point $\tilde{p} \in P$ is the smallest distance from $\tilde{p}$ to the nearest point on the boundary of P.

More generally, in the two-dimensional case, the *Radius of Stability* of a subset S of $\mathbb{R}^2$ at $\tilde{s} \in S$ is the radius of the largest circle centered at $\tilde{s}$ that is contained in S.

Hence, informally:

> The **radius of stability** of system q at the nominal state $s^* \in S_{stable}(q)$ is the radius of the largest *ball*[a] centered at s^* all of whose points are stable.
>
> [a]See short discussion on the concept "ball" in the Appendix A.

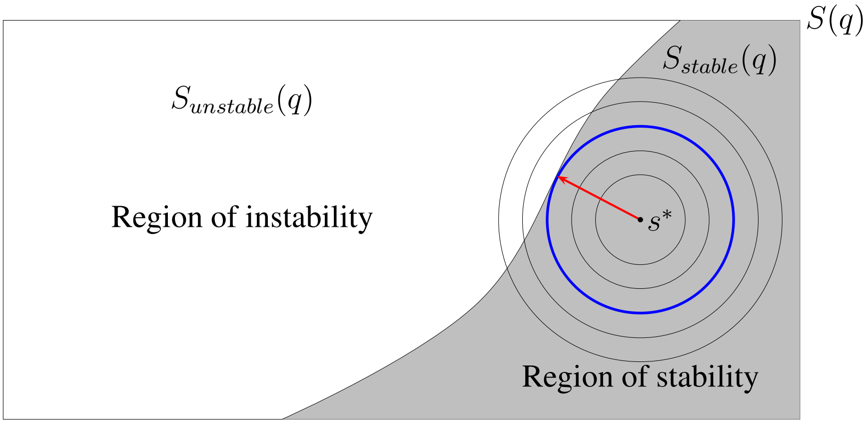


Figure 2.4: Radius of Stability at s^*

That is, it is the radius of the largest *ball* centered at s^* that is contained in the region of stability $S_{stable}(q)$.

This is illustrated in Figure 2.4.

It goes without saying that the *Radius of Stability* has many other (equivalent) interpretations. For example, as illustrated in Figure 2.4, the *Radius of Stability* of $S(q)$ at s^* is the distance of s^* from the *nearest* point on the boundary separating $S_{stable}(q)$ and $S_{unstable}(q)$. In the language of stability theory, the *Radius of Stability* is the *shortest* distance from the nominal state s^* to *instability*. It is the size of the *smallest perturbation* in the nominal state s^* that can destabilize the system.

Formally,

Definition 2.1.1 *Radius of stability of system $q \in Q$ at $s^* \in S_{stable}(q)$:*

$$\rho(q, s^*) := \max_{\rho \geq 0} \{\rho : s \in S_{stable}(q), \forall s \in B(\rho, s^*)\} \ , \ q \in Q \tag{2.3}$$

where

$$B(\rho, s^*) := \text{ ball of radius } \rho \text{ centered at } s^* \in S_{stable}(q). \tag{2.4}$$

Needless to say, the larger $\rho(q, s^*)$, the more stable system q, in the neighborhood of the *nominal* state s^*.

The question is then:

> What is the relevance of the intuitive, well-established concept *Radius of Stability* to our discussion on info-gap decision theory?

As it turns out, the relevance is immediate and of the first importance:

Definition 2.1.2 *Consider the instance of the generic Radius of Stability model (2.3) whose regions of stability are as follows:*

$$S_{stable}(q) = \{s \in S(q) : r^* \leq r(q, s)\} \ , \ q \in Q \tag{2.5}$$

where

$$r^* = \textit{a given critical performance level.}$$
$$r(q,s) = \textit{performance level of system } q \textit{ given that the system is in state } s.$$

We shall refer to a Radius of Stability model of this type as IGRM[2].

To put it more explicitly, from a *Radius of Stability* perspective, *IGRM* is a *Radius of Stability* model of the following simple format:

Radius of stability of system q at state s^* a la IGRM:

$$\rho(q,s^*) := \max_{\rho \geq 0} \{\rho : r^* \leq r(q,s), \forall s \in B(\rho,s^*)\} \ , \ q \in Q \tag{2.6}$$

In this case, we shall refer to $\rho(q,s^*)$ as the *IGRM robustness of decision* q at s^*.

In words,

> The IGRM robustness of system q at s^* is the radius of the largest ball $B(\rho,s^*)$ such that the performance requirement $r^* \leq r(q,s)$ is satisfied for all the states in this ball.

Since *IGRM* is none other than info-gap's generic robustness model, the following question is inevitable:

> Considering that info-gap's robustness model is, by definition, a *Radius of Stability* model, hence a model of *local* robustness, meaning that it is suitable for handling *small perturbations* in the value of a given nominal state, on what grounds can it possibly be claimed that info-gap decision theory is a suitable tool for modeling, analyzing, and managing *severe uncertainty*?

To answer this question we need to examine more closely the terminology used by info-gap decision theory and the meaning that it ascribes to the concept *severe uncertainty*.

2.2 Info-gap decision theory

As indicated above, info-gap decision theory is based on a generic robustness model that is a simple *Radius of Stability model* of the type specified by (2.6). Therefore, as shown in the preceding analysis, insofar as robustness is concerned, all that this theory can do — methodologically speaking — is to seek decisions that are robust against *small perturbations* in a given nominal value of a parameter of interest.

In contrast, in the info-gap literature this model of robustness is used as a framework for the analysis and management of robustness against *severe uncertainty.* The connection to *severe uncertainty* is established in info-gap decision theory by regarding the *state* s as a parameter whose true value is *unknown,* indeed is subject to *severe uncertainty*.

[2] Shorthand for *Info-Gap Robustness Model.*

The correspondence between the *Radius of Stability* terminology outlined above and the terminology used by info-gap decision theory — and the associated notations — are summarized in Figure 2.5.

Stability theory	Info-gap decision theory
system, $q \in Q$	decision, $q \in Q$
state, s	uncertain parameter, u
state space, $S(q)$	uncertainty space, $\mathscr{U}$
nominal state, s^*	estimate, $\tilde{u}$
radius ρ	horizon of uncertainty, α
ball, $B(\rho, s^*)$	region of uncertainty, $U(\alpha, \tilde{u})$
stability requirement, $s \in S_{stable}(q)$	performance requirement, $r^* \leq r(q, u)$

Figure 2.5: Comparison of terminology and notation

And so, info-gap decision theory defines the robustness of decision $q \in Q$ as follows[3]:

Info-gap's robustness model:

$$\hat{\alpha}(q, \tilde{u}) := \max \{\alpha \geq 0 : r^* \leq r(q, u), \forall u \in U(\alpha, \tilde{u})\}\ ,\ q \in Q, \tilde{u} \in \mathscr{U} \tag{2.7}$$

In the language of info-gap decision theory:

> The (info-gap) robustness of decision $q \in Q$, denoted $\hat{\alpha}(q, \tilde{u})$, is the largest horizon of uncertainty, α, such that the performance requirement $r^* \leq r(q, u)$ is satisfied at every point u in $U(\alpha, \tilde{u})$.

And in the language of the *Radius of Stability* model:

> The (info-gap) robustness of decision $q \in Q$ is the size of the largest perturbation in $\tilde{u}$ that does not violate the performance requirement $r^* \leq r(q, u)$ for any u within the range of this perturbation.

This is shown in Figure 2.6.

Therefore, for the record:

Theorem 2.2.1
Info-gap's robustness model is a simple Radius of Stability model.

Proof. Info-gap's robustness model is the simple instance of the *Radius of Stability model* whose elements are specified in Figure 2.5. QED

Next, given that info-gap decision theory ranks decisions according to their robustness: the larger the robustness the better, it follows that the best (optimal) decision is that whose robust-

[3] Note that, like "balls", the regions of uncertainty $U(\alpha, \tilde{u}), \alpha \geq 0$, are assumed to be nested. That is, $U(0, \tilde{u}) = \{\tilde{u}\}$ and $U(\alpha, \tilde{u}) \subseteq U(\alpha + \varepsilon, \tilde{u}), \forall \alpha, \varepsilon \geq 0$.

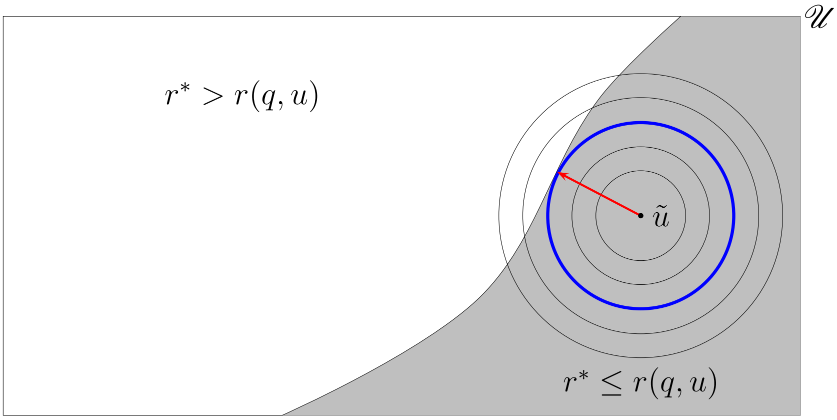


Figure 2.6: Info-gap's robustness of decision q at $\tilde{u}$

ness $\tilde{\alpha}(q,\tilde{u})$ is the largest over all $q \in Q$. This yields,

Info-gap's decision model:

$$\hat{\alpha}(\tilde{u}) := \max_{q\in Q} \hat{\alpha}(q,\tilde{u}) \tag{2.8}$$

$$= \max_{q\in Q} \max \{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{2.9}$$

In sum, in terms of stability theory, info-gap decision theory ranks decisions on grounds of their *Radius of Stability* at the point estimate $\tilde{u}$ with respect to a performance requirement of the form $r^* \leq r(q,u)$.

This conclusion brings out the two main issues that render info-gap decision theory problematic:

- Not only that info-gap's robustness model is not distinct, new, and radically different from mainstream models of robustness, it is in fact a simple instance of the most famous local stability (robustness) model: the *Radius of Stability model* (circa 1960).
- This model addresses the *local* robustness of decisions against **small perturbations**[4] in the value of the estimate $\tilde{u}$, **not** the *global* robustness of decisions against the severe uncertainty in the true value of u.

The first issue is far more serious than it may appear at first. The point is that the generic *Radius of Stability* model — hence info-gap's robustness model — is in fact a simple instance of *Wald's Maximin model* (circa 1940) — the bread and butter model of classical decision theory and robust optimization for the treatment of severe uncertainty (see Chapter 5).

For more than half a century this model has been used for the analysis and management of severe uncertainty. Since the 1970s, it has been one of the main tools used in the field of modern *robust optimization* (Rosenhead et al. 1972, Kouvelis and Yu 1997, Ben-Tal et al. 2009a).

The gravity of the second issue is due to info-gap decision theory's (mis)application of this model. Info-gap decision theory's declared objective is the pursuit of robustness to **severe**

[4]Recall that the *Radius of Stability* of q at $\tilde{u}$ is the size of the **smallest** perturbation in $\tilde{u}$ that can destabilize the system under consideration.

uncertainty. However, in the context of info-gap decision theory, this model is, as a matter of principle, applied to seek out decisions that are most robust (locally) in the neighborhood of the estimate $\tilde{u}$. It thus yields decisions that are not necessarily most robust (globally) against the *severe uncertainty* in the true value of u.

To illustrate this important point, consider Figure 2.7, where info-gap's robustness model is applied to a decision $q' \in Q$. Clearly, a quick comparison of Figure 2.6 and Figure 2.7 indicates that, according to info-gap decision theory, q is far more robust than q' in the neighborhood of $\tilde{u}$. But is q far more robust than q' against the *severe uncertainty* in the true value of u?

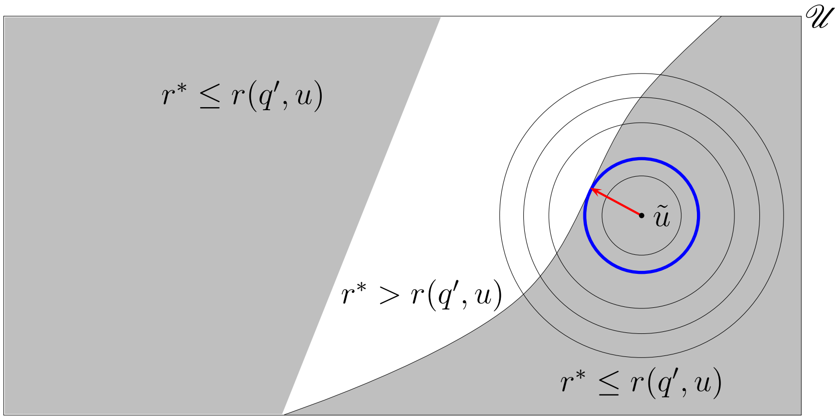


Figure 2.7: Info-gap's robustness of q' at $\tilde{u}$

The answer would depend, of course, on how we define (global) *robustness against severe uncertainty* in this case. With this in mind let us examine briefly what seems to be the most naturally obvious measure of global robustness.

2.3 Size criterion

This criterion ranks the robustness of a decision on grounds of the "size" of the subset of the uncertainty space on which the decision performs satisfactorily. The larger this subset, the more robust the decision. Thus, let

$$\mathscr{U}(q) := \{u \in \mathscr{U} \ : \ \textit{decision } q \textit{ performs satisfactorily at } u\} \ , \ q \in Q \tag{2.10}$$

We shall refer to $\mathscr{U}(q)$ as the *set of acceptable values* of u associated with decision q.

For example, in the context of info-gap decision theory, we have

$$\mathscr{U}(q) := \{u \in \mathscr{U} : r^* \leq r(q, u)\} \ , \ q \in Q \tag{2.11}$$

Next, define the (global) robustness of decision q against severe uncertainty as follows:

$$\gamma(q) := \frac{size(\mathscr{U}(q))}{size(\mathscr{U})} \ , \ q \in Q \tag{2.12}$$

where $size(A)$ denotes the "size" of set A. In words,

$\gamma(q)$ is the "fraction" of the uncertainty space $\mathscr{U}$ that satisfies the performance requirement. If $\gamma(q) = 1$ then q is super-robust, namely q *satisfies* the performance requirement over the entire uncertainty space $\mathscr{U}$. If $\gamma(q) = 0$, then q is super-fragile, namely it *violates* the performance requirement over the entire uncertainty space $\mathscr{U}$. If $\gamma(q) = 0.25$ then q satisfies the performance requirement $r^* \leq r(q, u)$ over 25% of the uncertainty space $\mathscr{U}$.

The point to note about $size(A)$ is that, if the sets under considerations are *discrete*, then we can let $size(A) = |A|$, where $|A|$ denotes the *cardinality* of set A. And in our two-dimensional case, the size of a set can be defined as the "area" of the set[5].

We shall refer to this intuitive measure of global robustness as the *Size Criterion*. The idea here is that, since the size of $\mathscr{U}$ is positive and it is independent of q, this criterion ranks decisions according to $size(\mathscr{U}(q))$: the larger $size(\mathscr{U}(q))$ the more robust q. Hence, if the objective is to select the most (globally) robust decision, the decision rule is as follows:

Size Criterion:
Rank decisions according to their $size(\mathscr{U}(q))$ values, hence select the decision whose $size(\mathscr{U}(q))$ value is the largest.
Thus, based on this criterion, the (global) robustness problem is as follows:

$$z^* := \max_{q \in Q} size(\mathscr{U}(q)) \tag{2.13}$$

In a word, this measure of (global) robustness to severe uncertainty entails that decision q' in Figure 2.7 is far more robust than decision q in Figure 2.6 against the severe uncertainty in the true value of u. It is important to note that according to this measure of robustness, the estimate $\tilde{u}$ has no impact whatsoever in determining the global robustness of decisions[6].

What makes info-gap's robustness model a model of *local* robustness is the fact that, like all *Radius of Stability* models, it seeks robustness against *small perturbations* in the value of the estimate $\tilde{u}$. For this reason it is vital to examine the role that the estimate $\tilde{u}$ plays in info-gap decision theory. This in turn requires a careful examination of how "severe uncertainty" is grasped, described and quantified in info-gap decision theory.

2.4 Threat problem revisited

At first glance, the connection between stability, *Radius of Stability*, and our *Threat Problem* seems rather natural. Each plan can be regarded as a "system", and a plan's "safe" region can be viewed as its region of stability.

In this case the *Radius of Stability* model would be used as a measure of the local robustness of each plan at any specified "nominal" location of the threat against small perturbations in the nominal location.

[5]For simplicity assume that the definition of "size" can handle uncountable sets.

[6]The above definition of robustness against severe uncertainty can be modified so that points in the vicinity of the estimate will have more "weight" than points that are more distant from the estimate

But is this what we should do? Is the *Radius of Stability* an appropriate measure of robustness in the context of the *Threat problem?*

The answer to this question depends, of course, on the "quality" of the information available to us on the location of the threat. To be precise, suppose that we have an *estimate* of the location of the threat, call it $\widehat{L}$. We can thus distinguish between the following cases regarding the "quality" of $\widehat{L}$:

- $\widehat{L}$ is perfect: we know that it is the exact (true) location of the threat.
- $\widehat{L}$ is a not a bad estimate, still it is questionable.
- $\widehat{L}$ is a wild guess based on rumors and gut feeling.

It goes without saying that if we are certain that $\widehat{L}$ is "perfect", then there is no need to conduct a *Radius of Stability* analysis, or for that matter, any other robustness analysis. Because, if $\widehat{L}$ is in the "safe" region of a certain plan, then this plan is certain to meet the threat successfully; and if $\widehat{L}$ is not in the "safe" region of a certain plan, then this plan will not meet the threat successfully.

All the same, having the value of the *Radius of Stability* in the neighborhood of $\widehat{L}$ can be useful in ranking the plans.

Next, consider the case where $\widehat{L}$ is a wild guess based on "rumors and gut feeling". Is the *Radius of Stability* at $\widehat{L}$ a suitable measure of robustness against the *severe uncertainty* in the true location?

For instance, consider Figure 2.8, where the radii of stability (ρ) of the six plans at a given wild guess $\widehat{L}$ are shown.

Note that it is clear that the *local robustness* of Plan F at $\widehat{L}$ is equal to 0. But does this mean that this plan has zero robustness to the *severe uncertainty* in the (unknown) true location of the threat? Surely, assigning zero robustness to severe uncertainty to Plan F and positive robustness to Plan A, B, C, and E is at odds with the fact that the region of stability of Plan F is larger than the regions of stability of Plan A, B, C, and E.

The inference is then that to deal with such issues we need to examine the meaning of the term "severe uncertainty" in this context and the role of an "estimate" in an analysis that seeks decisions that are robust against "severe" uncertainty.

The first issue requiring attention is the assumption that an estimate *exists*. This is important because the *Radius of Stability* model, hence info-gap's robustness model, **requires** exactly *one* nominal value (point estimate) of the parameter of interest. So the question is:

> What happens in situations where no estimate exists?
>
> Furthermore, what should be done in situations where more than one estimate is available?

The first questions may refer to situations where the quality of the estimate we have is so poor that it is deemed appropriate to "ignore" it. The second question refers to the commonly encountered situation where estimates provided by various methods/experts/sources are greatly at odds with one another.

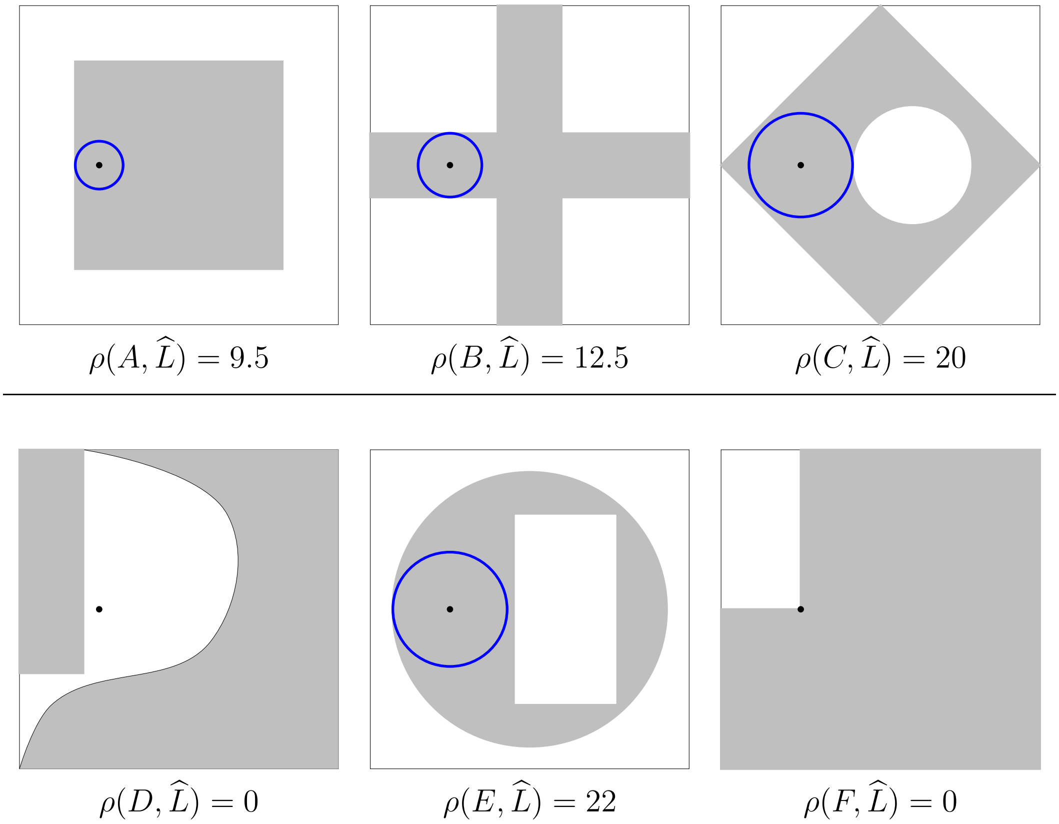


Figure 2.8: Radii of stability (ρ) of the plans at $\widehat{L}$ (small black dot)

To illustrate, consider the situation shown in Figure 2.9, where two estimates of the location of the threats are mooted. These are the two black dots in the uncertainty spaces of the six plans. The corresponding *Radii of Stability* are also shown.

How would we determine which plan is the most robust against the severe uncertainty in the true location of the threat?

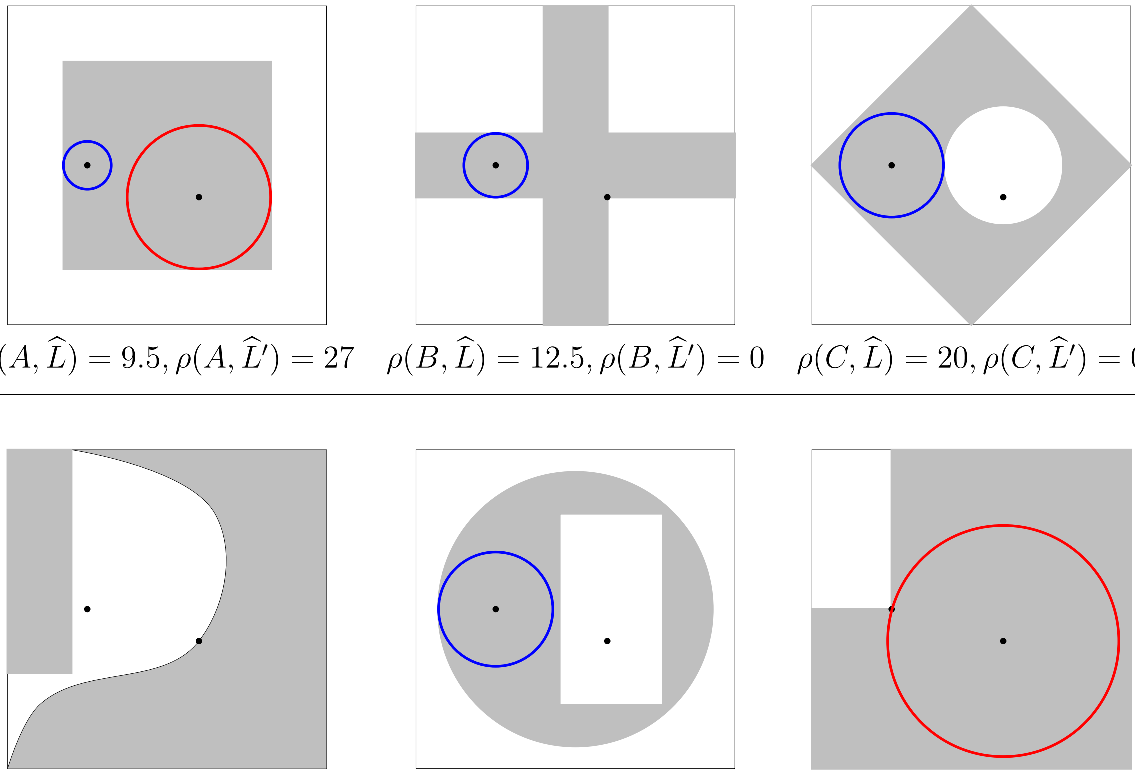
We can, of course, repeat the exercise with 3 estimates, 4 estimates, and so on.

In a nutshell, given that the uncertainty postulated by info-gap decision theory is *severe*, dealing with questions such as these is of the utmost importance as their implications are not only methodological. Their implications are immediate for the practical application of the theory.

2.5 Bibliographic notes

It is important to note that some of the intuitive measures of global robustness, such as the *Size Criterion*, were developed in the early days of *robust optimization* (e.g. Rosenhead et al. 1972, Rosenblat 1987, Kouvelis and Yu 1997).

However, these are rarely used in practice because they tend to give rise to extremely difficult optimization problems — problems that can be solved in practice only if certain simplifying conditions hold (e.g. Starr 1963, 1966, Schneller and Sphicas 1983, Eiselt and Langley 1990, Eiselt et al. 1998).



$\rho(A, \widehat{L}) = 9.5, \rho(A, \widehat{L}') = 27$ $\rho(B, \widehat{L}) = 12.5, \rho(B, \widehat{L}') = 0$ $\rho(C, \widehat{L}) = 20, \rho(C, \widehat{L}') = 0$

$\rho(D, \widehat{L}) = 0, \rho(D, \widehat{L}') = 0$ $\rho(E, \widehat{L}) = 22, \rho(E, \widehat{L}') = 0$ $\rho(F, \widehat{L}) = 0, \rho(F, \widehat{L}') = 44$

Figure 2.9: Radii of stability (ρ) of the plans at $\widehat{L}$ and $\widehat{L}'$ (small black dots)

As shown in Appendix C, in cases where the uncertainty space is discrete, the *Size Criterion* can be given a "classical" formulation using *Laplace's Principle of Insufficient Reason*. And as shown in Chapter 5, this criterion can be formulated as a *Maximin Rule.*

Details on the state of the art in *robust optimization* can be found in Ben-Tal et al. (2006, 2006a, 2009, 2009a).

There seems to be a general consensus in the area of *control theory* that the concept *Radius of Stability* was invented in the 1980s by Hinrichsen and Pritchard (1986a, 1986b). But a quick search of the literature suggests otherwise.

Indeed, earlier references to the concept *Radius of Stability* date back to the early 1960s. For example,

- Wilf H.S. (1960). Maximally stable numerical integration. *Journal of the Society for Industrial and Applied Mathematics,* 8(3):537-540.

- Milne W.E. and Reynolds R.R. (1962). Fifth-order methods for the numerical solution of ordinary differential equations. *Journal of the ACM,* 9(1):64-70.

In Milne and Reynolds (1962, p. 62) we read:

> It is convenient to use the term "radius of stability of a formula" for the radius of the largest circle with center at the origin in the s-plane inside which the formula remains stable.

The term was apparently coined independently by Hinrichsen and Pritchard (1986a, 1986b) in the field of *control theory.* So, according to Paice and Wirth (1998, p. 289):

> Robustness analysis has played a prominent role in the theory of linear systems. In particular the state-state approach via stability radii has received considerable attention, see [HP2], [HP3], and references therein. In this approach a perturbation structure is defined for a realization of the system, and the robustness of the system is identified with the norm of the smallest destabilizing perturbation. In recent years there has been a great deal of work done on extending these results to more general perturbation classes, see, for example, the survey paper [PD], and for recent results on stability radii with respect to real perturbations...

where HP2 = Hinrichsen and Pritchard (1990), HP3 = Hinrichsen and Pritchard (1992) and PD= Packard and Doyle (1993).

As for the role of *Radius of Stability* in *optimization* and *mathematical programming*, consider Zlobec's (1988, p. 129) statement:

> An important concept in the study of regions of stability is the "radius of stability" (e.g., [65, 89]). This is the radius r of the largest open sphere $S(\theta^*, r)$, centered at θ^*, with the property that the model is stable, at every point θ in $S(\theta^*, r)$. Knoweldge of this radius is important, because it tells us how far one can uniformly strain the system before it begins to "break down". (In an electrical power system, the latter may manifest in a sudden loss of power, or a short circuit, due to a too high consumer demand for energy. Our approach to optimality, via regions of stability, may also help understand the puzzling phenomenon of voltage collapse in electrical networks described, e.g., in [11].)

where [65] = Petric and Zlobec (1983), [89]= Zlobec (1987), and [11] = Carpentier et al (1984).

In the first edition of the *Encyclopedia of Optimization,* Zlobec (2001) describes the *Radius of stability* as follows:

> The radius of the largest ball centered at θ^*, with the property that the model is stable at its every interior point θ, is the radius of stability at θ^*, e.g, [69]. It is a measure of how much the system can be uniformly strained from θ^* before it starts breaking down.

where [69] = Zlobec (1988).

In accounting (Raab and Feroz, 2007, p. 400):

> Charnes, Haag, Jaska, and Semple (1992), Charnes, Rousseau, and Semple (1996) and Seiford and Zhu (1998) developed a sensitivity analysis technique based on the infinitynorm measure of a vector. This technique defines the necessary simultaneous perturbations to the component vector of a given NG that cause it to move to a state of "virtual" efficiency. Virtual efficiency is defined as a point on the efficient frontier where any miniscule detrimental perturbation (increase in inputs and/or decrease in outputs) will cause an efficient NG to become inefficient, or any miniscule favorable perturbation

> (decrease in inputs and/or increase in outputs) will cause an inefficient NG to become efficient.
>
> For an efficient NG, the infinity-norm measure, or the radius of stability (herein termed stability index), defines the largest "cell" in which all simultaneous detrimental perturbations to the input and output components will not cause a change in the efficiency status from technically efficient to inefficient. As such, the larger the stability index, the more robustly efficient the NG is said to be. Those efficient NGs with small stability indices will thus become technically inefficient, with smaller detrimental perturbations than those efficient NGs with larger stability indices.

NG = national government.

And here are the abstracts of two recent recent papers where the *Radius of Stability* plays a central role. First the paper "Finite cooperative games: parametrisation of the concept of equilibrium (from Pareto to Nash) and stability of the efficient situation in the Hölder metric" by Emelichev and Karlkina (2009, p. 229):

> We consider a finite cooperative game of several players with parametric principle of optimality such that the relations between players in a coalition are based on the Pareto maximum. The introduction of this principle allows us to find a link between such classical concepts as the Pareto optimality and the Nash equilibrium. We carry out a quantitative analysis of the stability of the game situation which is optimal for the given partition method with respect to perturbations of parameters of the payoff functions in the space with the Hölder l_p-metric, $1 \leq p \leq \infty$. We obtain a formula for the radius of stability for such situation, so we are able to point out the limiting level for perturbations of the game parameters such that the optimality of the situation is preserved.

Next, the paper entitled "Sensitivity and Stability Analysis in DEA on Interval Data by Using MOLP Methods" by Beigi et al. (2009, p. 891):

> In this paper, we suppose a method for analyzing sensitivity and stability of all the decision making units, while inputs and outputs are interval data. Therefore, for estimating radius of stability of a DMU; firstly, we classify the decision making units then we obtain the radius of stability for each classification. For analyzing the sensitivity and estimating the radius of stability analogous of each DMU, a MOLP is defined. Therefore, the interactive methods are used for finding the efficient solution in which the comment of Decision Maker is important. At the end numerical example has been solved by using the weighted-sums of the target function and also the interactive method (STEM) in MOLP problems.

And finally, consider this (emphasis added):

> In contrast to standard robust optimization approach, our focus in this paper is not a problem of finding a robust solution for a given set of scenarios (corresponding to some δ), but rather a question of the robustness of a solution being optimal for the initial

> weights. In particular, we are interested in the largest value of δ, for which this solution remains robust. Such a value of δ is called the **robustness radius** of the considered solution. Main results of this paper concern some lower bounds for this radius.
>
> Libura (2009, p. 672)

Note that this *robustness radius* is analogues to the radius of stability of the *optimal* solution. Libura (2010) considers radii of stability of this type, as well as measures of global of robustness analogous to the *Size Criterion*.

The following, much "older", phrasing of the *post-optimality* problem addressed by radius of stability models is interesting in that it makes no reference to uncertainty. Rather, the problem is phased as a "perturbation" problem:

> A new direction in combinatorial optimization, connected with stability analysis of the solution, is reviewed. The major part of the paper deals with problems like the following one. Given solution t (or the whole solution set) of a discrete optimization problem Z, stability analysis consists in finding an answer to the question: *By how much can we perturb numerical parameters of the problem Z without loss of the property of t to be optimal (respectively, without extending the solution set)?*
>
> Sotskov et al. (1995, p. 169)

Note that both Sotskov et al. and (1995) Libura (2009, 2010) deal with *combinatorial* optimization problems.

Remark

Considering the wide-ranging use of the concept *Radius of Stability* as a measure of local robustness/stability, the lack of all reference to this concept in the info-gap literature (e.g. Ben-Haim 2001, 2006, 2010) is incomprehensible. But more than this, given that info-gap's robustness model is in fact a simple *Radius of Stability* model, the claims in the info-gap literature (e.g. Ben-Haim 2001, 2006) that info-gap decision theory is radically different from all current theories for decision under uncertainty are without any foundation.

In Chapter 7 I elaborate on these, and related, issues.

Chapter 3

Severe uncertainty

As indicated in the previous chapter, the uncertainty model that info-gap decision theory puts forward for the modeling, analysis, and management of severe uncertainty consists of three key ingredients:

$$
\begin{aligned}
u &= \text{ parameter of interest.} \\
\mathscr{U} &= \text{ set of all possible/plausible values of } u. \\
\tilde{u} &= \text{ point estimate of the true value of } u.
\end{aligned}
$$

In the next section I discuss briefly the assumptions that info-gap decision theory posits about these objects, and I explain how these assumptions give meaning to the *severity* of the uncertainty postulated by info-gap decision theory.

3.1 Working assumptions

Although the term *severe uncertainty* is not given a formal *definition* in the info-gap literature, the working assumptions posited about it make it quite clear how *severe uncertainty* is understood in info-gap decision theory:

- The (point) estimate $\tilde{u}$ is a **poor** indication of the true value of u, indeed it is likely to be substantially **wrong.**
- The uncertainty space $\mathscr{U}$ can be **vast**, even **unbounded.**
- The uncertainty model is **non-probabilistic** and **likelihood-free.**

The third item implies that info-gap decision theory does not postulate any assumptions about the likelihood of the true value of u being in any one particular neighborhood of the uncertainty space $\mathscr{U}$. To further clarify this "likelihood-free" assumption, consider the situation depicted in Figure 3.1 where the rectangle represents the uncertainty space $\mathscr{U}$ and A and B are two arbitrary values of u.

If the uncertainty model is *likelihood-free,* then there are no grounds to assume that the true value of u is more/less likely to be in any one particular neighborhood of $\mathscr{U}$. Thus, there are no grounds to assume that the true value of u is more/less likely to be in the neighborhood of

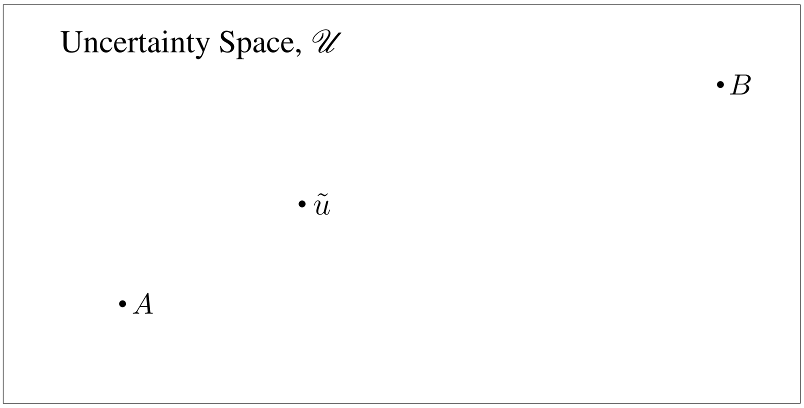


Figure 3.1: Severe uncertainty

$u = A$ rather than in the neighborhood of $u = B$. Specifically, there are no grounds to assume that the true value of u is more/less likely to be in the neighborhood of the estimate $\tilde{u}$ rather than in the neighborhood of any other point in $\mathscr{U}$, say A or B.

It should be noted that positing the above working assumptions to convey the meaning of the term *severe uncertainty* is thoroughly in line with the manner in which this concept is understood in other decision theories. In particular, *severe uncertainty* is conceived of in a roughly similar manner in *classical decision theory* (Resnik 1987, French 1988) and in *robust optimization* (e.g. Kouvelis and Yu 1997, Ben-Tal et al. 2009).

And it should also be pointed out that a likelihood-free quantification of uncertainty is attributed to the *prima facie* milder term "uncertainty", in the wider literature. This is the norm, for instance in undergraduate textbooks on *Operations Research* (e.g. Winston 2004, Hillier and Lieberman 2005). I discuss the taxonomy of the concept *uncertainty* in Section 3.9.

At this stage, it suffices to say that in this discussion I accept the above working assumptions for what they are. Namely, as . . . working assumptions posited by info-gap decision theory to describe the severity of the uncertainty under consideration. I take it for granted that users of this theory are aware of their responsibility to ascertain that the problem that they are concerned with is indeed represented adequately by these assumptions.

3.2 Neighborhood structure

Recall that info-gap decision theory postulates a *neighborhood* structure on the uncertainty space $\mathscr{U}$. This structure consists of *balls (neighborhoods)* $U(\alpha, \tilde{u}), \alpha \geq 0$, centered at the (point) estimate $\tilde{u}$. These balls are required to satisfy two axioms:

$$\textbf{Contraction: } U(0, \tilde{u}) = \{\tilde{u}\} \tag{3.1}$$

$$\textbf{Nesting: } U(\alpha, \tilde{u}) \subseteq U(\alpha + \varepsilon, \tilde{u}) \ , \ \forall \alpha, \varepsilon \geq 0 \tag{3.2}$$

It is extremely important to keep in mind that this is precisely where the likelihood-free property comes into play. The balls, constituting neighborhoods in the info-gap uncertainty model, do not in any way shape or form give expression to a likelihood structure on $\mathscr{U}$. The points in $U(\alpha, \tilde{u})$ are no more than points in $\mathscr{U}$ that are within a "distance" of α from the estimate $\tilde{u}$.

And what is more, this "distance" is never specified a-priori in the specification of the decision problem itself. The neighborhoods are not objects representing properties or features of the problem considered. They are constructs that info-gap decision theory itself imposes on $\mathscr{U}$ for the purpose of measuring robustness.

Another point that one must never lose sight of is the centrality of the point estimate $\tilde{u}$ in info-gap decision theory. In sharp contrast to mainstream decision theories dealing with severe uncertainty, where using an estimate is not even contemplated, in info-gap decision theory the estimate $\tilde{u}$ is the fulcrum of the robustness model. This estimate — as born out by info-gap publications, for instance Ben-Haim (2007) — is regarded a "guess", indeed sometimes no more than a "wild guess".

Still, as pointed out above, this characterization of the estimate as giving content to the concept *severe uncertainty* is fully in line with the general understanding of *severe uncertainty* in the literature on robust optimization (e.g. Kouvelis and Yu 1997, Ben-Tal et al. 2009) and classical decision theory (e.g. Resnik 1997, French 1988). But, it is precisely because the estimate is assumed to be so poor that it is . . . **not** incorporated formally (explicitly) in the uncertainty models of robust optimization and classical decision theory, in cases where the uncertainty is severe.

Having said all that, the following question is inevitable:

> Given that the estimate may well be no more than a "wild guess", that the uncertainty space can be vast (even unbounded) and that the uncertainty model as a whole is likelihood-free, on what grounds can we take the local robustness in the neighborhood of the point estimate to be a reliable indication of the global robustness against severe uncertainty?

The answer to this is obvious:

> It is an established fundamental, indeed one that can be easily illustrated (as done above), that generally, *local* robustness does not imply *global* robustness, and vice versa. Hence, any claim that local robustness implies global robustness must be justified, or proved.

This takes us back to the role of a poor point estimate (wild guess) in decision-making under severe uncertainty in general, and to its role in models of local robustness in particular.

3.3 Threat problem revisited

What implications do the working assumptions discussed above have for our preliminary analysis of the *Threat problem?*

The most obvious one is that we cannot, more precisely, we have no grounds whatsoever to single out the neighborhood of the estimate $\widehat{L}$ for a *local* robustness analysis as an "approximation" of a *global* robustness analysis over the entire uncertainty space. In particular, arguments such as "the estimate $\widehat{L}$ is the most likely value of the true location of the threat" are groundless. First, we must assume that the estimate is poor, so that it can be no more than a wild guess. Second, the uncertainty model is "likelihood-free", so there is no reason to believe that the true location of the threat in most likely to be in the neighborhood of the estimate. In short, there is no reason to believe that the true value is in the neighborhood of the estimate.

The question therefore arises: in the context of a likelihood-free model of uncertainty, in what way, if any, is the estimate $\widehat{L}$ distinguishable from other locations in the metropolitan area? In what way can we treat it differently from the other possible locations?

3.4 Size of the uncertainty space

The severity of the uncertainty in the true value of the parameter of interest is manifested not only in the poor quality of the estimate but also in the **size** of the uncertainty space under consideration.

This is illustrated in Figure 3.2 where two uncertainty spaces are shown for the same system. Note that the two uncertainty spaces have the same point estimates, $\tilde{u}' = \tilde{u}'' = \tilde{u}$, and that these are of similar quality. Both are say, *wild guesses.* The difference is in the **size** of the uncertainty space: $\mathscr{U}''$ is much larger than $\mathscr{U}'$.

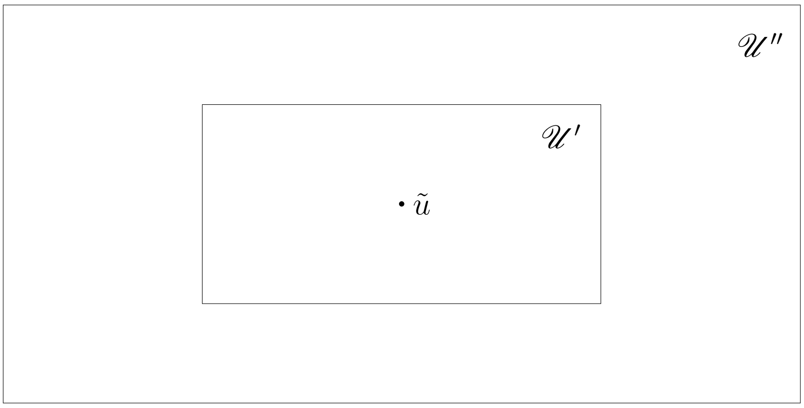


Figure 3.2: Two uncertainty spaces for the same system

This may represent a situation where $\mathscr{U}'$ is an initial approximation of the uncertainty space that figures in a preliminary analysis of the problem, whereas $\mathscr{U}''$ is the uncertainty space in the final analysis. The change occurred as additional information about the problem became available.

Clearly, the uncertainty associated with $\mathscr{U}''$ is *more severe* than the uncertainty associated with $\mathscr{U}'$. So the following practical question arises:

How do we determine the (size of the) uncertainty space $\mathscr{U}$?

We can distinguish between two cases:

- *Bounded* uncertainty spaces.
- *Unbounded* uncertainty spaces.

Note that although in the latter we do not have to worry about how the boundaries of the uncertainty space should be determined — as there are no boundaries — it is nevertheless incumbent on us to justify the choice of the model as an adequate representation the problem under consideration.

The point in discussing these obvious issues is to call attention to the fact that local robustness models, such as the *Radius of Stability* model, are not designed to deal with large perturbations (large uncertainty spaces), hence they are thoroughly unsuitable for handling situations where the severity of the uncertainty is manifested in a large uncertainty space. It is therefore important to understand the basic issues involved in determining the "size" of the uncertainty space $\mathscr{U}$.

The best way to make this point vivid is to show that *Radius of Stability* models are invariant with the "size" of the uncertainty space $\mathscr{U}$ when this size is greater than the *Radius of Stability* of the system under consideration. I refer to this properly as the *Invariance Property* (Sniedovich 2007, 2010, 2011).

3.5 Invariance property of radius of stability models

To illustrate the self-evident fact captured by the *Invariance Property*, consider the *Radius of Stability* of $\mathscr{U}'$ shown in Figure 3.3, where as above, the shaded area represents the points satisfying the performance requirement under consideration.

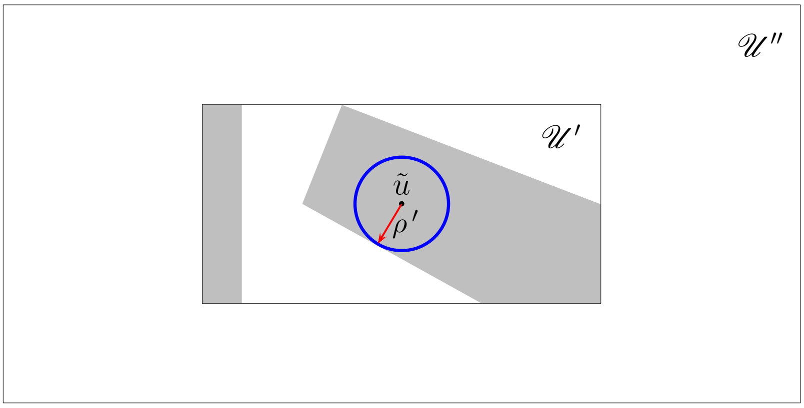


Figure 3.3: Radius of stability of $\mathscr{U}'$

Next, consider the *Radius of Stability* of the larger uncertainty space $\mathscr{U}''$ that contains $\mathscr{U}'$. Clearly, there is no need to perform the (local) Radius of Stability robustness analysis for $\mathscr{U}''$ as it will yield the same results (radius of stability) as those yielded by the analysis of $\mathscr{U}'$, and this regardless of the shape of the "safe" region in the area of $\mathscr{U}''$ that is outside $\mathscr{U}'$ (not shown here).

Similarly, the *Radius of Stability,* ρ', will not change for a smaller $\mathscr{U}'$ so long as the uncertainty space contains the ball centered at $\tilde{u}$ whose radius is equal to the *Radius of Stability* of $\mathscr{U}'$, namely the ball $B(\rho', \tilde{u})$.

More formally, this basic property of the *Radius of Stability* can be described as follows:

Invariance Property:[a] The *Radius of Stability* is invariant with the “size” of the uncertainty space.
More precisely, let ρ' be the *Radius of Stability* of a system, and assume that for some $\rho^* > \rho'$, the ball $B(\rho^*, \tilde{u})$ is contained in the uncertainty space under consideration.
Then, the *Radius of Stability* will remain unchanged despite changes in the uncertainty space, so long as the uncertainty space contains the ball $B(\rho^*, \tilde{u})$.

[a]The need for the larger radius ρ^* arises only in some pathologic cases. Under regular conditions $\rho^* = \rho'$.

This is illustrated in Figure 3.4 where the solid rectangle represents the “original” uncertainty space with respect to which the *Radius of Stability* was determined. The dashed rectangles represent other uncertainty spaces for which the *Radius of Stability* remains the same.

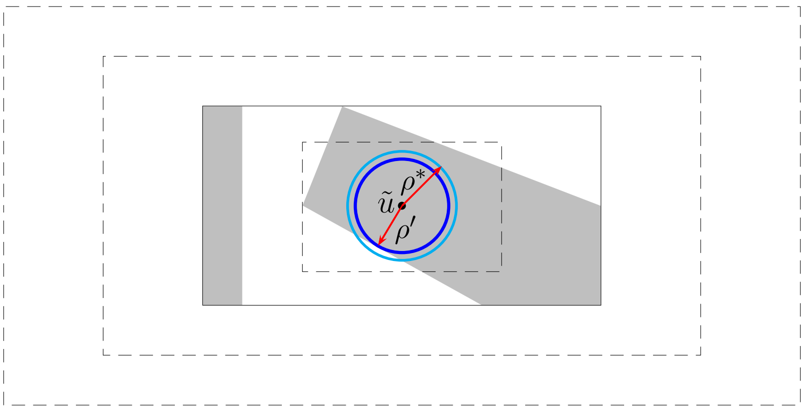


Figure 3.4: Invariance property of the *Radius of Stability*

What this property brings out then is that the *Radius of Stability* model does not (cannot) give sound representation to the size of the “safe” region in a large uncertainty space.

It should be stressed, however, that this property is not an issue in the conventional (intended) use of the *Radius of Stability,* namely in situations where it is used as a measure of the local robustness at a given nominal point. This property is problematic, methodologically and practically, only in situations where the *Radius of Stability* is (mis)used as a measure of *global* robustness against *severe uncertainty.*

3.6 Bounded spaces

If the uncertainty regarding the true value of the parameter u is indeed severe, determining the boundaries of the uncertainty space $\mathscr{U}$ accurately may prove difficult. This in turn may make it difficult to determine the global robustness of the system accurately/reliably. And to illustrate the basic issue here, consider the case where

u = number of kangaroos in Australia on January 1, 2020.

We can then let $\mathscr{U} = [\underline{u}, \overline{u}]$, where $\underline{u}$ and $\overline{u}$ are lower and upper bounds on u, respectively. For instance, how about letting $\underline{u} = 0$ and $\overline{u} = 200$ million?[1]

While it can be argued that these bounds are indeed bounds, they most definitely are not "realistic" (tight) bounds. Thus, if the results of the robustness analysis depend on these bounds, it will be wise to attempt to determine more "realistic" (tighter) bounds.

Since info-gap decision theory does not deal with this difficulty, I do not discuss it here, except to point out the following.

If a methodology that is designed to tackle global robustness against severe uncertainty generates results that are independent of the size (boundaries) of the uncertainty space, then this should immediately raise the alarm of an imminent *Invariance Property*. This in turn should immediately alert one to the possibility that the methodology is in all likelihood based on a model of local — rather than global — robustness.

3.7 Unbounded uncertainty spaces

Since robust decision-making under severe uncertainty often requires dealing with *rare events, catastrophes and surprises*, methodologies for decision-making under severe uncertainty are expected to be able to handle large uncertainty spaces. So it is not surprising that according to Ben-Haim (2006, p. 210, emphasis is added):

> **Most** of the commonly encountered info-gap models are **unbounded.**

But, as shown in the preceding chapter, info-gap's robustness model is a (local) *Radius of Stability* model, so the following question arises:

> How can the **local** *Radius of Stability* model deployed by info-gap decision theory as a robustness model possibly take on **unbounded** uncertainty spaces?

The answer to this important question is given by the *Invariance Property:*

> Methodologically speaking, info-gap's local robustness model does not, indeed cannot, tackle unbounded uncertainty spaces. This is so because this model is, by definition, oblivious to the behavior of the performance requirement outside the largest safe ball around the point estimate. Namely, it takes no account whatsoever of the area outside the ball centered at the point estimate whose radius is equal to the distance between the point estimate and the nearest point in the uncertainty space that violates the performance requirement.

This is shown in Figure 3.5, where the bold curve is the boundary between the "safe" and "unsafe" regions in the neighborhood of the estimate $\tilde{u}$, and u° denotes the nearest point to the estimate on this curve. Note that neither the uncertainty region nor the boundary are completely specified. For simplicity assume that $\mathscr{U}$ is unbounded. This means that the segment of the boundary (as demonstrated by this figure) is **infinitesimally small** relative to the entire boundary.

[1] The present population is about 60 million, give or take a couple of millions.

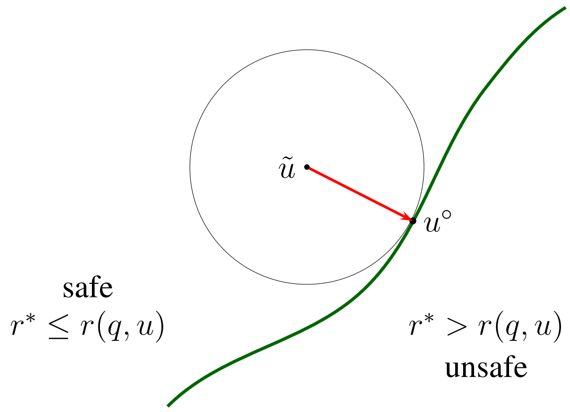


Figure 3.5: Invariance property of *Radius of Stability* models

Clearly, this type of analysis completely ignores the behavior of the system in regions that are distant from $\tilde{u}$ and u°, therefore — methodologically speaking — it is ill-suited for the treatment of unbounded uncertainty spaces.

A more edifying illustration of this effect of the *Radius of Stability* model can be given in the case where the uncertainty space $\mathscr{U}$ is the real line, as shown in Figure 3.6.

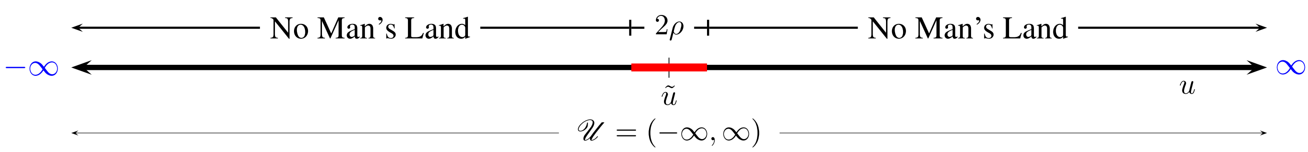


Figure 3.6: The *No Man's Land* property of *Radius of Stability* models

The two intervals called *No Man's Land* represent the subsets of the uncertainty space $\mathscr{U}$ that are "ignored" by the *Radius of Stability* model. If $\mathscr{U}$ is unbounded, these sections cover practically the entire uncertainty space. If we let $U(\rho, \tilde{u}) := \{u \in \mathscr{U} : |u - \tilde{u}| \leq \rho\}$, then the neighborhood of $\mathscr{U}$ that determines the *Radius of Stability* of decision q, represented as the thick red line segment, whose width is 2ρ, where ρ is equal to the *Radius of Stability* of decision q.

This representation of the local analysis conducted by the *Radius of Stability* model raises this rhetorical question:

> Can an analysis of a minute (indeed, an infinitesimally small) segment of the unbounded uncertainty space be claimed, let alone be expected to, determine the robustness of the system against the severe uncertainty in the true value of the parameter?

The obvious answer is:

> The *Radius of Stability* model does not seek to determine the robustness of the system against severe uncertainty in the true value of the parameter.
>
> By virtue of its design, all it can do is seek to determine the local robustness of the system against small perturbations in a given value (estimate) of the parameter of interest.

> To apply it as a tool for determining the robustness of the system against severe uncertainty in the true value of the parameter of interest amounts to a misapplication of the concept "radius of stability".

This observation is valid not only in the framework of unbounded uncertainty spaces. That is, it is valid in cases where the uncertainty space is much larger than the largest safe ball associated with the *Radius of Stability* model.

3.8 Rare events, catastrophes, and surprises

As indicated above, an important factor that is expected to enter the analysis in robust decision-making under severe uncertainty, is that of *rare events, catastrophes and surprises*. This is acknowledged in the info-gap literature

> It is the rare events — catastrophes, for example — which are often of greatest concern to the decision maker.
>
> Ben-Haim (2006, p. 18)

> Rare events in probabilistic models models are described by the tails of the distribution, while probability distributions are usually specified in terms of mean and mean-deviation parameters. This makes probabilistic models risky design tools , since it is rare events, the catastrophic ones, which must underlie the reliable design.
>
> Ben-Haim (2006, pp. 330-331)

> The management of surprises to the "economic problem", and info-gap theory is a response to this challenge. This book is about how to formulate and evaluate economic decisions under severe uncertainty. The book demonstrates, through numerous examples, the info-gap methodology for reliably managing uncertainty in economic policy analysis and decision making.
>
> Ben-Haim (2010, p. x)

It follows then that our assessment of info-gap decision theory will have to be based, among other things, on its ability to reliably manage the modeling and analysis of *rare events, catastrophes* and *surprises.*

However, given that info-gap's robustness model is a *Radius of Stability* model, the question obviously is:

> How can a (local) *Radius of Stability* model reliably model and analyze the impact of rare events, catastrophes and surprises in decision-making in the face of severe uncertainty?

And the answer obviously is: **it can't!**

3.9 Classification of uncertainty

It is accepted practice in *classical decision theory* to distinguish among three states of affairs:

- Certainty
- Risk
- Uncertainty

The first represents situations where the “true” values of all the parameters of a decision model are known. The second represents situations where, although the true values of the model’s parameters are unknown, it is possible to use probabilistic models to quantify the uncertainty pertaining to the true values of these parameters. The third represents situations where the uncertainty in the unknown true values of the parameters cannot be quantified by probabilistic models.

The distinction between “Risk" and “Uncertainty" is due to the economist Frank Knight (1885 – 1972), hence the popular phrase *Knightian uncertainty* used to designate an uncertainty that cannot be quantified by probabilistic/likelihood models.

Since the phrase *Knightian uncertainty* is often used in the info-gap literature to convey the type of uncertainty that info-gap decision theory is claimed to model, analyze and manage, it is important to point out the following.

Knight’s (1921) own elaboration of the difference between “risk” and “uncertainty” suggests that he might have had a more potent, more general conception of uncertainty in mind — one that also encompasses what we call today “unknown unknowns”. Here is Knight’s phrasing of this point:

> To preserve the distinction which has been drawn in the last chapter between the measurable uncertainty and an unmeasurable one we may use the term “risk” to designate the former and the term “uncertainty” for the latter. The word “risk” is ordinarily used in a loose way to refer to any sort of uncertainty viewed from the standpoint of the unfavorable contingency, and the term “uncertainty” similarly with reference to the favorable outcome; we speak of the “risk” of a loss, the “uncertainty” of a gain. But if our reasoning so far is at all correct, there is a fatal ambiguity in these terms, which must be gotten rid of, and the use of the term “risk” in connection with the measurable uncertainties or probabilities of insurance gives some justification for specializing the terms as just indicated. We can also employ the terms “objective” and “subjective” probability to designate the risk and uncertainty respectively, as these expressions are already in general use with a signification akin to that proposed. The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation a priori or from statistics of past experience), while in the case of uncertainty this is not true, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique. The best example of uncertainty is in connection with the exercise of judgment or the formation of those opinions as to the future course of events, which opinions (and not scientific knowledge) actually guide

> most of our conduct.
>
> Knight (1921, III.VIII.1-2)

The assertion of particular interest is:

> ... the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique ...

This statement seems to suggest that in Knight's understanding, "uncertainty" also refers to states of affairs where our lack of knowledge impedes not only the ability to specify the probabilities of events of interest, but also the *uncertainty space* of the problem. So it is not only the probabilities that are difficult to quantify, it is the events themselves that are elusive.

John Maynard Keynes' (1883 – 1946) phrasing of the distinction also seems to point in this direction:

> By "uncertain" knowledge, let me explain, I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty; nor is the prospect of a Victory bond being drawn. Or, again, the expectation of life is only slightly uncertain. Even the weather is only moderately uncertain. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, or the obsolescence of a new invention, or the position of private wealth owners in the social system in 1970. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know. Nevertheless, the necessity for action and for decision compels us as practical men to do our best to overlook this awkward fact and to behave exactly as we should if we had behind us a good Benthamite calculation of a series of prospective advantages and disadvantages, each multiplied by its appropriate probability, waiting to be summed.
>
> Keynes (1937, pp. 213-214)

Over the years a host of descriptors have been added to the term "uncertainty" in an ongoing effort to capture the "true nature" of uncertainty and to distinguish perhaps between various degrees or levels of magnitude thereof. To mention just a few:

> Strict uncertainty, Severe uncertainty, Extreme uncertainty, Deep uncertainty, Substantial uncertainty, Essential uncertainty, Hard uncertainty, High uncertainty, True uncertainty, Fundamental uncertainty, Wild uncertainty, Radical uncertainty, Profound uncertainty, Knightian uncertainty, True Knightian uncertainty.

In a recent paper[2], Lo and Mueller (2010) put forward the idea that there is a spectrum ranging from certainty to various "levels" of uncertainty. They propose a taxonomy of uncertainty that is far more refined: "... one capable of explaining the differences across the entire spectrum of intellectual pursuits from physics to biology to economics to philosophy and religion ...". Thus, the spectrum that they propose consists of the following levels:

[2] I thank Bob Williamson (NICTA) for bringing this paper to my attention at the end of October 2010.

- Level 1: *complete certainty.*
  An idealized deterministic world.
- Level 2: *risk without uncertainty.*
  The randomness under consideration is governed by a probabilistic model and the set of possible outcomes is completely known.
- Level 3: *fully reducible uncertainty.*
  This is a weaker version of risk in that it " ...can be rendered arbitrarily close to Level-2 uncertainty with sufficiently large amounts of data using the tools of statistical analysis. "
- Level 4: *partially reducible uncertainty.*
  Here "...there is a limit to what we can deduce about the underlying phenomenon generating the data." Consequently, in this environment "...classical statistics may not be as useful as a Bayesian perspective, in which probabilities are no longer tied to relative frequencies of repeated trials, but now represent degree of belief".
- Level 5: *irreducible uncertainty.*
  This represents *total ignorance*, that is " ...ignorance that cannot be remedied by collecting more data, using more sophisticated methods of statistical inference or more powerful computers, or thinking harder and smarter."
- Level ∞: *Zen uncertainty.*
  "Attempts to understand uncertainty are mere illusions; there is only suffering."

Considering that info-gap decision theory is supposed to take on situations that are not amenable to probabilistic models (classical or Bayesian), its uncertainty model should by right fall under *Level 5: Irreducible uncertainty.* It is instructive therefore to quote the entire paragraph from Lo and Mueller (2010, p. 13) where this level of uncertainty is described in some detail:

> Irreducible uncertainty is the polite term for a state of total ignorance; ignorance that cannot be remedied by collecting more data, using more sophisticated methods of statistical inference or more powerful computers, or thinking harder and smarter. Such uncertainty is beyond the reach of probabilistic reasoning, statistical inference, and any meaningful quantification. This type of uncertainty is the domain of philosophers and religious leaders, who focus on not only the unknown, but the unknowable.
>
> Stated in such stark terms, irreducible uncertainty seems more likely to be the exception rather than the rule. After all, what kinds of phenomena are completely impervious to quantitative analysis, other than the deepest theological conundrums? The usefulness of this concept is precisely in its extremity. By defining a category of uncertainty that cannot be reduced to any quantifiable risk — essentially an admission of intellectual defeat — we force ourselves to stretch our imaginations to their absolute limits before relegating any phenomenon to this level.

The inference to be drawn from this portrayal of *irreducible uncertainty* is that the uncertainty that info-gap decision theory concerns itself with is actually of the *Level 4: partially reducible uncertainty* type, rather than of the *Level 5: irreducible uncertainty* type. And this implies, in turn, that Bayesian models of uncertainty can be used to quantify the uncertainty dealt with by info-gap decision theory. Indeed, as indicated by Hansen and Sargent (2010, p. 3, emphasis added):

> After Knight (1921), Savage (1954) contributed an axiomatic treatment of decisionmaking in which preferences over gambles could be represented by maximizing expected utility defined in terms of subjective probabilities. Savage's work extended the earlier justification of expected utility by von Neumann and Morgenstern (1944) that had assumed known objective probabilities. Savage's axioms justify subjective assignments of probabilities. Even when accurate probabilities, such as the fiftyfifty put on the sides of a fair coin, are not available, decision makers conforming to Savage's axioms behave as if they form probabilities subjectively. **Savage's axioms seem to undermine Knight's distinction between risk and uncertainty.**

In fact, as we shall see in the ensuing sections, the argument that one would have to make to justify the use of an info-gap robustness model would inevitably land one in "Bayesian territory" . This is so because, to account for the centrality of the *point estimate* $\tilde{u}$ in info-gap's robustness analysis, and to make a case for the inherently local analysis conducted around this point estimate, one would have to impose certain additional assumptions on the uncertainty model. These assumptions would effectively amount to attributing a *likelihood structure* to the uncertainty space $\mathscr{U}$. And this fact provides further proof that Bayesian models of uncertainty can handle (would be more adept at handling?) the (purportedly) non-probabilistic problems handled by info-gap decision theory.

3.10 Role of point estimates in likelihood-free models

As indicated earlier, there are models for decision under severe uncertainty — notably Wald's Maximin model and its many variates, as well as Laplace's Model of Insufficient Reason (Resnik 1987, French 1988) — that do not require a point estimate of the true value of the parameter of interest. This is hardly surprising, considering that models of severe uncertainty are — by definition — non-probabilistic and likelihood-free.

Indeed, methodologically speaking, it is either pointless or extremely difficult to incorporate a point estimate of the true value of the parameter of interest in a truly likelihood-free model. To see why this is so, consider the two cases depicted in Figure 3.7 where the *Radius of Stability* is determined for system q twice.

Suppose that the story behind these two cases is as follows.

- Case 1 represents a preliminary analysis of the system, at which stage the estimate $\tilde{u}$ is taken to be a "rough wild guess" of the true value of u.
- Case 2 represents the final analysis of the system after it was concluded that $\tilde{u}$ — the same estimate used in Case 1 — in fact proved a pretty good estimate.

Since $\tilde{u}$ has the same value in both cases, the *Radius of Stability* is also the same in both cases. The implication is that the results generated by the model are independent of the "quality" of

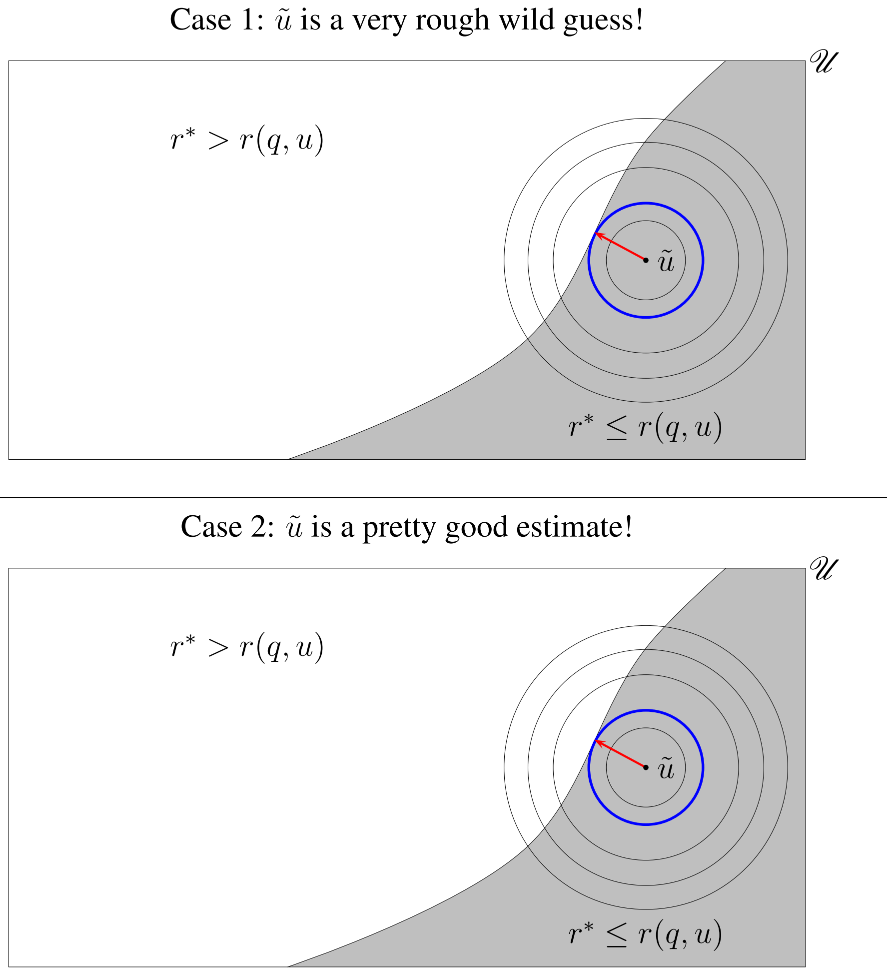


Figure 3.7: Two *Radius of Stability* models

$\tilde{u}$, they depend only on its value.

But, the point is that the "quality" of the estimate $\tilde{u}$ is crucial for determining our "confidence" in the results generated by the model in the sense that the "better" the estimate, the more "confident" we are that the local robustness is a good approximation of the robustness against the uncertainty in the true value of u.

So, what all this boils down to is that a total incongruity exists between the severity of the uncertainty under consideration and the quality of the estimate required to justify a local, *Radius of Stability* type of robustness analysis. This is how Rout et al. (2009) allude to this contradiction within the framework of info-gap decision theory:

> Thus, the method challenges us to question our belief in the nominal estimate, so that we evaluate whether differences within the horizon of uncertainty are 'plausible'. Our uncertainty should not be so severe that a reasonable nominal estimate cannot be selected.
>
> Rout et al. (2009, p. 785)

In other words, the idea is that for the local robustness analysis conducted by info-gap decision theory in the neighborhood of the estimate to be compatible with the stated aim of the pursuit of decisions that are robust to uncertainty, the estimate must be 'reasonable', or differently put, the uncertainty should not be too severe.

But the trouble is, as indicated below, that translating such qualitative considerations into a strictly likelihood-free quantitative model of uncertainty is a well-nigh impossible task.

3.11 Imported likelihood

An obvious way to justify the use of a local robustness model as a means for approximating global robustness against severe uncertainty is to "import" likelihood into the uncertainty model. The idea here is that in so doing one would presumably be able to contend that the true value of u is "quite likely" to be in the neighborhood of the estimate $\tilde{u}$, hence focusing the robustness analysis on this neighborhood would make sense.

More formally, consider the ball $B(\rho^*, \tilde{u})$, where ρ^* is the *Radius of Stability* of the system at $\tilde{u}$. If it can be shown that the true value of u is very likely to be in $B(\rho^*, \tilde{u})$, then we would be justified in arguing that ρ^* is a suitable measure of robustness against the uncertainty in the true value of u.

To the best of my knowledge, the first attempt to "import" likelihood of this kind into an info-gap uncertainty model was made by Hall and Harvey's (2009, p. 2), where the following additional condition was imposed on the standard likelihood-free info-gap uncertainty model:

> An assumption remains that values of u become increasingly unlikely as they diverge from $\tilde{u}$.

But the rub is that this assumption is too weak to justify the local robustness analysis in the neighborhood of $\tilde{u}$, as it still falls short of assuring that the true value of u is indeed "quite likely" to be in the neighborhood of the estimate $\tilde{u}$. A simple counter example will suffice.

Consider the case where the uncertainty space is the real-line and u is the realization of a *normally distributed random variable.* Let $\tilde{u}$ be the *mean* of the distribution. Clearly, this distribution satisfies Hall and Harvey's (2009) assumption.

However, by increasing the *variance* of the distribution as required, for any $\delta > 0$ we can significantly decrease the probability that the true value of u is in the interval $[\tilde{u} - \delta, \tilde{u} + \delta]$, as shown in Figure 3.8.

Thus, to provide the assurance that the true value of u is quite likely to be in the neighborhood of $\tilde{u}$, we must also require the *variance of the distribution to be sufficiently small.*

And the trouble with this is that "importing" such assumptions into the model throws the severe uncertainty — as postulated by info-gap decision theory — out of the window.

It should be noted that this is not a technical issue that can be remedied by some quick fix: it is a fundamental difficulty that boils down to this.

> Methodologically speaking, in decision-making under severe uncertainty, unless some conditions hold, there is no assurance that a local robustness analysis in the neighborhood of a poor estimate is a good approximation of global robustness. Claims to the contrary must be justified, or proved or in the very least argued for — they cannot just be made.

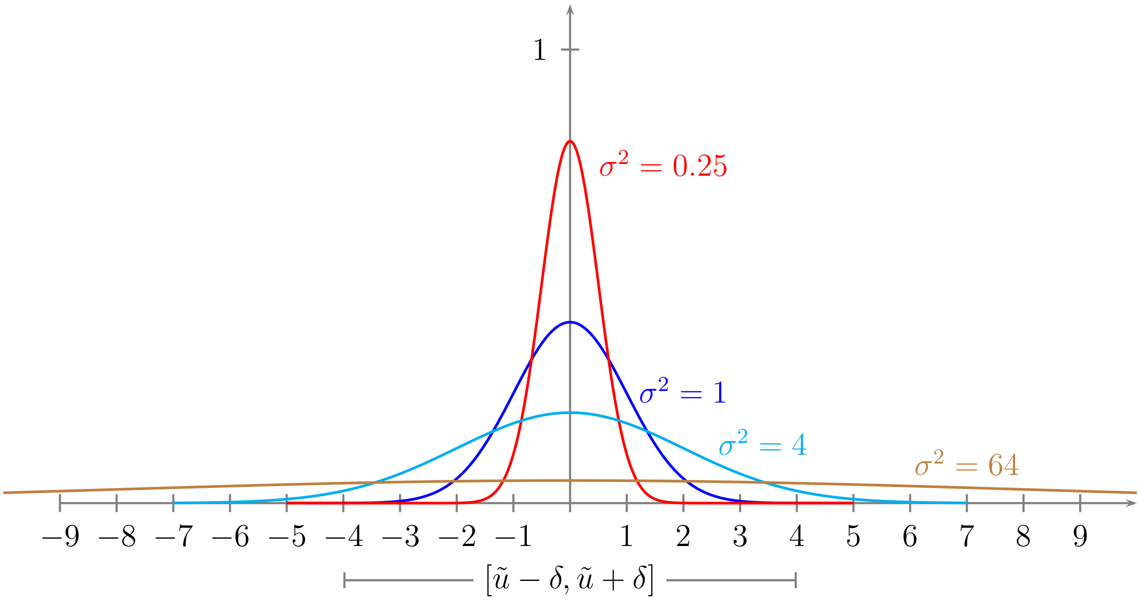


Figure 3.8: Normal distributions, mean = $\tilde{u} = 0$

This observation also applies to the most recent quick-fix proposed by Hine and Hall (2010, pp. 2-3):

> The main assumption is that u, albeit uncertain, will to some extent be clustered around some central estimate $\tilde{u}$ in the way described by $U(\alpha, \tilde{u})$, though the size of the cluster (the horizon of uncertainty α) is unknown. In other words, there is no known or meaningfully bounded worst case. Specification of the info gap uncertainty model $U(\alpha, \tilde{u})$ may be based upon current observations or best future projections.

In short, in cases where there is a justification for imposing a likelihood structure on the uncertainty space, this must be done formally, through the front door, and not surreptitiously, through the back door.

See discussion on related issues in Appendix D and Appendix E.

3.12 Threat problem revisited

To appreciate the difficulties encountered in incorporating an estimate in a likelihood-free uncertainty model, consider Figure 1.1, and assume that no estimate is available. Which plan is the most robust against the severe uncertainty in the true location of the threat?

Next, consider Figure 2.8, keep in mind that the model is likelihood-free and address the following two questions:

- Which is the most robust plan in the neighborhood of the estimate?
- Which is the most robust plan against severe uncertainty?

You may want to answer these questions by way of repeatedly varying the "quality" of the estimate from say, "excellent", to "very rough wild guess".

Then, consider Figure 2.9 and repeat the exercise, this time with the two point estimates.

3.13 An applied ecology perspective

Having elucidated the working assumptions that info-gap decision theory makes about the uncertainty that it aims to manage and the technical issues associated with these assumptions, let us examine briefly how the severity of the uncertainty is perceived by info-gap scholars, say in the field of applied ecology.

Figure 3.9, reconstructed from a figure provided in Halpern et al. (2006, p. 3), illustrates the view that info-gap scholars in the field of applied ecology have of the theory's position vis-a-vis other theories/approaches for dealing with uncertainty. Note that info-gap decision theory is viewed as a method designed for the most exacting uncertainty: the uncertainty space is **unbounded** and the quantification of uncertainty is **non-probabilistic** and **likelihood-free.**

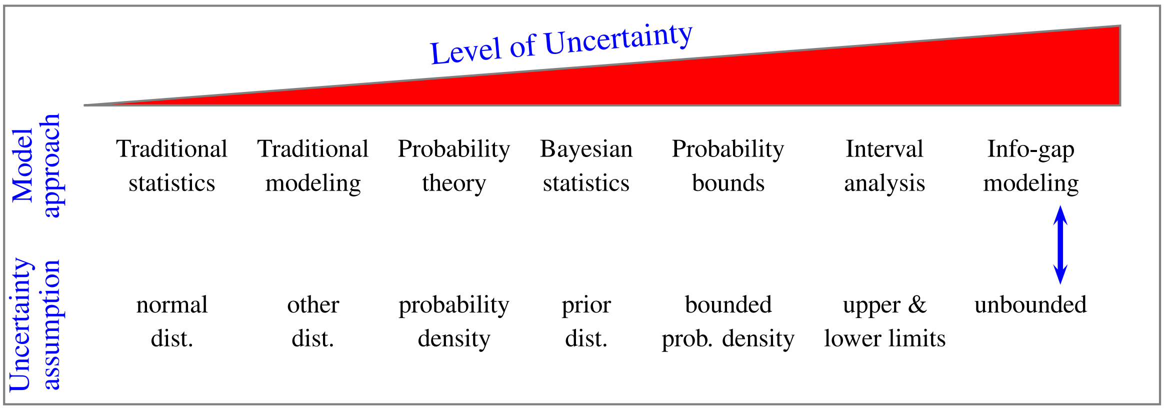


Figure 3.9: Treatment of various levels of uncertainty

This figure, which leaves no room for debate about the perception (in applied ecology) as to the role and place of info-gap decision theory in the management of severe uncertainty, is consistent with views expressed elsewhere in the applied ecology, conservation biology and environmental management literatures.

There are two radically different ways of looking at this fact:

either

· the perception expressed in this figure is consistent with the character and capabilities of info-gap decision theory (Ben-Haim 2001, 2006, 2010),

or

· the perception expressed in this figure is based on a misinterpretation of the robustness analysis conducted by info-gap decision theory.

As indicated at the outset, the main goal of this discussion is to point out that the latter is the case and to explain in detail how this misconception arises.

Based on the discussion in the preceding chapters, it is pretty much obvious that this misconception stems from a lack of appreciation of the difference between *local* and *global* robustness, coupled with a lack of appreciation of the type of robustness analysis prescribed by info-gap decision theory. For, as we saw above, info-gap's inherently *local* robustness analysis, renders it utterly unsuitable for the management of severe uncertainty which calls for a *global* approach to robustness.

Remark

I should point out that for all the misconceptions about info-gap decision theory's treatment of severe uncertainty, exhibited in Figure 3.9, there is no suggestion in it that info-gap decision theory is capable of handling ... "unknown unknowns". It is therefore surprising that info-gap decision theory had recently been proposed for this very purpose (e.g. Wintle et al. 2010, Wintle 2010) in the applied ecology literature (see Chapter 7).

Chapter 4

Worst-case analysis

Worst-case analysis is used for a variety of purposes. In this discussion I examine it as a means for dealing with *variability*, including variability induced by *uncertainty*. The objective of this discussion is to elucidate certain conceptual and technical issues that are related to the worst-case analysis that is prescribed by info-gap decision theory. My principal aim is to clarify the difference between *local* and *global* worst-case analysis.

This is a key issue in our discussion.

First, worst-case analysis is at the heart of both a local and a global robustness analysis, hence it plays a pivotal role in robust decision-making and robust optimization. Second, this important tool of thought is badly misrepresented in the info-gap literature.

To illustrate the latter point, note that the info-gap literature is spotted with statements denying that info-gap's robustness analysis is a worst-case analysis. Indeed, Ben-Haim (2001, 2006, 2010) is adament that info-gap's robustness analysis is **not** a worst-case analysis. These assertions are intended to support the (erroneous) claim that info-gap's robustness model is not a Maximin model. For instance (emphasis added):

> **Optimization of the robustness in eq. (3.172) is emphatically** *not* **a worst-case analysis.** In classical worst-case min-max analysis the decision maker minimizes the impact of the maximally damaging case. But an info-gap model of uncertainty is an unbounded family of nested sets: $U(\alpha, \tilde{u})$, for all $\alpha \geq 0$. Consequently, there usually is no worst case: any adverse occurrence is less damaging than some other more extreme event occurring at a larger value of α.
>
> Ben-Haim (2006, p. 101)

> **Info-gap theory is not a worst-case analysis.** While there may be a worst case, one cannot know what it is and one should not base one's policy upon guesses of what it might be. Info-gap theory is related to robust-control and min-max methods, but nonetheless different from them. The strategy advocated here is not the amelioration of purportedly worst cases.
>
> Ben-Haim (2010, p. 9)

So one of the main objectives of the discussion in this chapter is to show formally that, contrary to Ben-Haim's numerous claims, info-gap's robustness analysis is indeed a typical (local) worst-case analysis.

However, before I can turn to the main business of this chapter, it is important to point out that when it comes to the management of *uncertainty,* worst-case analysis is not necessarily a measure of last resort. Namely, one does not turn to a worst-case analysis only in pathologic cases where probabilistic models cannot be used for lack of data and/or knowledge.

Indeed, in many cases worst-case analysis supplements, rather than supplants, probabilistic methods. That is, there are many cases where we are interested not only in say, the *expected value* of the outcome, but also in the worst outcome.

The following (famous) example illustrates a case where the same problem has a probabilistic version and a worst-case version.

4.1 Example: The counterfeit coin problem

You are given a collection of N coins all of which, except one, have the same weight. Your task is to identify the odd coin, using a balance scale. For simplicity assume that the odd coin is heavier than the other coins.

Figure 4.1: Counterfeit coin problem

Consider the first weighing: you place x coins on each side of the scale and $N - 2x$ coins are left off the scale. Since you have no clue where the odd coin is, you cannot predict the result of the weighing. Hence, there is an uncertainty in the result of the first weighing.

The implication is then that the number of weighings required to identify the odd coin cannot be determined a priori even if the weighing strategy is stipulated in detail a priori. There is a strong element of "chance" here.

To deal with the uncertainty (which coin is the odd one), we can develop a probabilistic model to determine the results of the weighings. For instance, we can assume that the probability that the odd coin is on the scale is equal to the proportion of the number of coins placed on the scale (relative to the total number of coins yet to be inspected). Thus, if for instance, there are 80 coins altogether and 30 are placed on each side of the scale, then the probability that the odd coin is on the scale would be equal to 60/80. This means that the weighing has two possible outcomes: the number of coins left for inspection after the weighing will be either 30 or 20 with probability 0.75 and 0.25, respectively.

Proceeding on this approach, we can then set up an optimization model that will require the optimal solution to minimize the *expected value* of the number of weighings required to identify the odd coin (e.g. Sniedovich 2003).

Alternatively, we may want to find a weighing policy that is best under the *worst-case scenario.* That is, a policy that minimizes the number of weighings required to identify the odd coin assuming that Nature (Chance) is "playing against us". Note that if the assumption is that Nature (Chance) is playing against us, then in each weighing the odd coin will be taken to be hidden in the *largest* of the three batches of coins: the two on the scale and the one off the scale. This means that Nature's antagonistic attitude enables us to predict with certainty Her policy, to thereby remove the uncertainty regarding the result of a weighing altogether.

For example, if in the first weighing we place x coins on each side of the scale, then the result of the weighing will be as follows:

· If $x \geq N - 2x$ then x coins will be left for inspection.

· If $x < N - 2x$ then $N - 2$ coins will be left for inspection.

More compactly, $\max(x, N - 2x)$ coins will be left for inspection after the first weighing.

Given this observation, we can set up an optimization model that requires the optimal weighing policy to minimize the number of weighings required to identify the odd coin under the worst-case scenario (e.g. Sniedovich 2003).

4.2 Global worst-case analysis

Suppose that the performance of a system $q \in Q$ depends on some parameter $u \in \mathscr{U}$, where $\mathscr{U}$ denotes the set of possible values of u. We shall refer to $\mathscr{U}$ as the *parameter space*.

For example, suppose that the performance of q is measured by its ability to satisfice a performance requirement $r^* \leq r(q, u)$, where r^* is a given critical level of performance and $r(q, u)$ denotes the performance level of q given u.

Recall that this is precisely the situation that we discussed at the outset in relation to decision-making under uncertainty. The difference is that here we do not assume that the variability in the value of u is due to uncertainty. For instance, it can be due to a completely deterministic, controlled, variation of the parameter u.

Now, typically, for some values of u in $\mathscr{U}$ system q satisfices the constraint $r^* \leq r(q, u)$, while for others it does not. So the question is: how should the performance of q be evaluated, given the variation in the value of u?

If we want, or are required, to evaluate the performance of system q under the *worst-case scenario,* we evaluate its performance by identifying the worst element of $\mathscr{U}$ relative to the *performance criterion.* In other words, under the worst-case scenario the performance of q will be determined with respect to the least favorable $u \in \mathscr{U}$ insofar as the requirement $r^* \leq r(q, u)$ is concerned.

Note that if the performance of q is measured by the value of $r(q, u)$ and the *larger* this value, the better, then the worst u in $\mathscr{U}$ is a $u \in \mathscr{U}$ that *minimizes* $r(q, u)$ over $\mathscr{U}$. If, on the other hand, the performance of q is measured by the value of $r(q, u)$ and the *smaller* this value, the better, then the worst u in $\mathscr{U}$ is a $u \in \mathscr{U}$ that *maximizes* $r(q,u)$ over $\mathscr{U}$.

We shall refer to worst-case analysis of this type as **global**, meaning that the evaluation of q is based on the worst value of u in the parameter space $\mathscr{U}$.

Needless to say, to apply this approach, a worst case must exist. As we shall see, though, in the case of info-gap decision theory, the existence of a worst case is not an issue. Nevertheless, because questions regarding the existence of a worst case are raised frequently in the info-gap literature (e.g. Ben-Haim 2001, 2006, 2010), it is important to give this matter the attention that it deserves.

4.2.1 Existence of a worst case

There are of course pathologic cases where a worst $u \in \mathscr{U}$ does not exist. For instance, consider the case where $Q = [20, 30]$, the parameter space is $\mathscr{U} = [0, 1) := \{u : 0 \leq u < 1\}$ and the performance function is specified by $r(q,u) = 100 - qu$, assuming that the larger $r(q,u)$ is, the better. Since, for any given $q \in Q$ the value of $r(q,u) = 100 - qu$ is decreasing with u, the worst u in $\mathscr{U}$ with respect to decision q is the largest element of $\mathscr{U}$. But, since $\mathscr{U}$ does not contain a "largest" element (note that the end point 1 is not an element of $\mathscr{U}$), there is no worst case here: for every value of $u \in \mathscr{U}$ there is an even worse value in $\mathscr{U}$.

Technically, such pathologic cases are associated with problems where the parameter space $\mathscr{U}$ is not closed[1].

A more dramatic case involves **unbounded** parameter spaces. For instance, consider the case where $Q = [20, 30]$, the parameter space is $\mathscr{U} = [0, \infty)$ and the performance criterion is specified by $r(q,u) = 100 + qu$, assuming that the smaller $r(q,u)$ is the better. Since $r(q,u)$ is increasing with u, the worst $u \in \mathscr{U}$ is the largest element of $\mathscr{U}$. However, $\mathscr{U}$ is unbounded above, hence there is no worst case here.

But the point to note here is that this example is **no proof** that in situations where $\mathscr{U}$ is unbounded a worst case does not exist as a matter of principle. Indeed, consider for instance the case where $Q = [0, 1]$, the parameter space is $\mathscr{U} = (-\infty, \infty)$ and the performance criterion is specified by $r(q,u) = q + \sin(u)$, assuming that the smaller $r(q,u)$ is, the better. Since $r(q,u)$ is increasing with $sin(u)$, the worst value of $u \in \mathscr{U}$ is that which maximizes $\sin(u)$ over $\mathscr{U}$. Thus, we conclude that the worst value of u is any value of $u \in \mathscr{U}$ such that $\sin(u) = 1$, and there are infinitely many (countable) such worst values (see Figure 4.2).

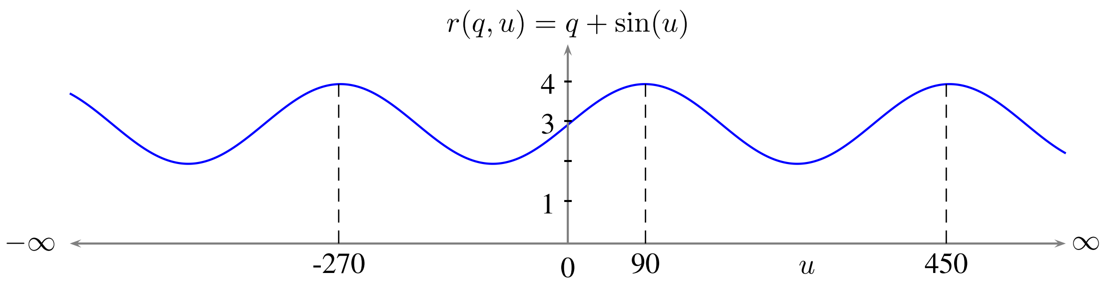


Figure 4.2: Worst cases within an unbounded parameter space, $q = 3$

[1] Roughly, a *closed* set is a set that contains its boundary. It is the complement of an *open* set.

Still, it is important to be clear on what exactly is the issue here.

It is no doubt true that in the absence of any assumptions about $\mathscr{U}$ and the performance function r, there is no assurance that a worst case with respect to $r(q,u)$ exists on $\mathscr{U}$.

But the whole point is that the question whether or not a worst case exists in the context of **info-gap decision theory** must be answered **within the framework of info-gap decision theory**. And in this framework, it is important to note, the worst-case analysis prescribed by info-gap's robustness model must be conducted with respect to the performance *requirement* $r^* \leq r(q,u)$, not the performance *level* $r(q,u)$. Because, it is with respect to this performance requirement that info-gap's robustness is determined.

And the upshot of all this is that, much as info-gap's robustness model does not conduct a global worst-case analysis on $\mathscr{U}$, a global worst-case with respect to the performance requirement $r^* \leq r(q,u)$ **always exists** even if $\mathscr{U}$ is unbounded.

This is so because insofar as this requirement is concerned, there are **only two possible outcomes**: either the requirement is **satisfied,** or it is **violated.** Hence, **there is always a worst case.**

More specifically, consider a decision $q \in Q$ and the requirement $r^* \leq r(q,u)$. We can distinguish between the following three possible cases:

- Case 1: $r^* \leq r(q,u), \forall u \in \mathscr{U}$.
  Each $u \in \mathscr{U}$ is both a worst case and a best case.
- Case 2: $r^* > r(q,u), \forall u \in \mathscr{U}$.
  Each $u \in \mathscr{U}$ is both a worst case and a best case.
- Case 3: $r^* > r(q,u)$ for some $u \in \mathscr{U}$ and $r^* \leq r(q,u)$ for some $u \in \mathscr{U}$.
  Each $u \in \mathscr{U}$ such that $r^* > r(q,u)$ is a worst case, and each $u \in \mathscr{U}$ such that $r^* \leq r(q,u)$ is a best case.

To draw an analogy with *optimization theory,* consider the optimization problem

$$\max_{x \in X} f(x) \tag{4.1}$$

where X is some set and f is a real valued function on X. For simplicity assume that f attains a global maximum and a global minimum on X.

Case 1 and Case 2 are analogous to a situation where $f(x)$ does not vary with $x \in X$, that is, situations where $f(x) = f(x') = \text{constant}$ for all $x, x' \in X$. In this case all the points in X are global maxima and global minima. Case 3 represents situations where there is some variation in $f(x)$ over $x \in X$, hence the global minima and global maxima are distinct.

What is important to keep in mind then is the distinction between

- a worst case with respect to the **constraint** $r^* \leq r(q,u)$;

and

- a worst case with respect to the **performance level** $r(q,u)$.

Thus, a worst case with respect to $r^* \leq r(q,u)$ always exists, whereas a worst case with respect to $r(q,u)$ may, or may not, exist.

For example, suppose that $Q = [100, 200]$, $\mathscr{U} = (-\infty, \infty)$, $r^* = 20$, and $r(q, u) = |q - 150| - u$. Then clearly, Case 3 above applies for each $q \in Q$. In fact, for each $q \in Q$ there are infinitely many distinct worst cases with respect to $r^* \leq r(q, u)$ in $\mathscr{U}$.

Now, for a $q \in Q$, consider the worst case of $r(q, u)$, assuming that the larger $r(q, u)$, the better. In this context the worst case for $r(q, u)$ is a $u \in \mathscr{U}$ that minimizes the value of $r(q, u)$ over $u \in \mathscr{U}$. But clearly, no such worst case exists because $r(q, u) = |q - 150| - u$ decreases linearly with u and $\mathscr{U}$ is unbounded (below).

In sum: because info-gap's robustness model is concerned with a "constraint", a worst case always exists, and the issue of the existence of a worst case is therefore a non-issue. Hence, statements such as this, that can be found in abundance in the info-gap literature, are at best misleading:

> In many cases the uncertainty is also unbounded, meaning that we do not know the worst case for u.
>
> Davidovitch et al. (2009, p. 2788)

They refer to "worst case with respect to $r(q, u)$", not to "worst case with respect to $r^* \leq r(q, u)$", even though *info-gap robustness* is defined in terms of worst case with respect to $r^* \leq r(q, u)$.

But more importantly: the worst-case analysis conducted by info-gap decision theory is **not** a global worst case analysis.

4.3 Local worst-case analysis

There are many situations where the decision maker has control not only of the decision variable $q \in Q$, but also of that part of the parameter space $\mathscr{U}$ that should be considered in the analysis. Symbolically, let y denote the parameter that the decision maker utilizes for this purpose, and let $\mathscr{U}(y)$ denote the subset of $\mathscr{U}$ associated with the value of y selected by the decision maker.

In practice, $\mathscr{U}(y)$ may represent the subset of $\mathscr{U}$ whose elements are relevant to the performance analysis of decision q, the point being that not all the elements of $\mathscr{U}$ are necessarily relevant to all the decisions in Q. It can also provide the decision maker with a mechanism for controlling the level of variability (risk) that can be incorporated in the worst-case analysis.

In short, using this parameter the decision maker can fine-tune the worst-case analysis to suit her needs. Since formally, both q and y are controlled by the decision maker, they are both "decision variables", and therefore with no loss of generality we can assume that y is actually incorporated in q. However, for the purposes of this discussion it is convenient and instructive to treat q and y as two separate, distinct objects such that $y \in Y(q)$, where for each $q \in Q$, $Y(q)$ is some given set specifying the set of values of y associated with decision q.

In this framework, the worst case associated with decision q is the worst u in $\mathscr{U}(y)$ rather than the worst u in $\mathscr{U}$, where y is the value selected by the decision maker in conjunction with q. We shall refer to such a worst case analysis as a *partial* worst-case analysis: it is *partial* because it is not conducted on the entire parameter space $\mathscr{U}$, but rather on a *subset* thereof.

A *local* worst-case analysis is a partial worst-case analysis where the set $\mathscr{U}(y)$ is a *neighborhood* of some point in $\mathscr{U}$ and the size (radius) of this neighborhood is determined by y. For example, $\mathscr{U}(y)$ could be a ball whose radius and center point are determined by y. In fact, y can be a pair $(\alpha, \tilde{u})$ where α specifies the radius of the ball and $\tilde{u}$ specifies its center point, so that $\mathscr{U}(y) = \mathscr{U}(\alpha, \tilde{u})$ denotes a ball of radius α around $\tilde{u}$.

With the aid of such a device, the decision maker can control the "level" or "degree" of the worst-case analysis: the larger the radius (α), the more severe the worst-case analysis is.

As indicated above, in practice, the parameter y can be incorporated in the decision variable q. In fact, we can view y as an integral part of q. This is consistent with the role of y in the worst-case analysis. It is a tool that enables the decision maker to control the parameter space so that each decision $q \in Q$ can have its own parameter space, and furthermore, this parameter space can be controlled by the decision maker. Clearly, this modeling device can be used to great effect in a worst-case analysis.

4.3.1 A Radius of Stability perspective

Recall that the *Radius of Stability* of system q is the *radius* of the largest ball centered at s^* all of whose elements are stable, and that the larger the radius is the better:

$$\rho(q, s^*) := \max_{\rho \geq 0} \{\rho : s \in S_{stable}(q), \forall s \in B(\rho, s^*)\} \ , \ q \in Q \tag{4.2}$$

where $B(\rho, s^*)$ denotes a ball of radius ρ around s^* and $S_{stable}(q)$ denotes the set of stable states associated with system q.

The fact that this analysis is a local worst-case analysis is broadcast by the $\forall s \in B(\rho, s^*)$ clause. Because, what this clause connotes is that to decide whether a given value of ρ is admissible, we need to establish whether the *least favorable* s in the ball $B(\rho, s^*)$ is stable. If it is stable, then ρ is admissible; if it is not, then ρ is not admissible (it is too large).

Assuming that $s^* \in S_{stable}(q), \forall q \in Q$, we can distinguish between the following two cases:

- Case A: $s \in S_{stable}(q), \forall s \in B(\rho, s^*)$.
  Each $s \in B(\rho, s^*)$ is both a worst case and a best case (on $B(\rho, s^*)$).
- Case B: $s \in S_{stable}(q)$ for some, but not all, $s \in B(\rho, s^*)$.
  Each $s \in B(\rho, s^*)$ such that $s \in S_{stable}(q)$ is a best case (on $B(\rho, s^*)$), and each $s \in S(q)$ such that $s \notin B(\rho, s^*)$ is worst case (on $B(\rho, s^*)$).

This is illustrated in Figure 4.3. For ρ', each element of $B(\rho', s^*)$ is both a best case and a worst case (on $B(\rho', s^*)$). In contrast, for ρ'', each $s \in B(\rho'', s^*)$ such that $s \in S_{stable}(q)$ is a best case (on $B(\rho'', s^*)$), and each $s \in S(q)$ such that $s \notin B(\rho'', s^*)$ is worst case (on $B(\rho'', s^*)$).

In short, the generic *Radius of Stability* model is a local worst-case model par excellence, where the radius ρ plays the role of the parameter y alluded to above.

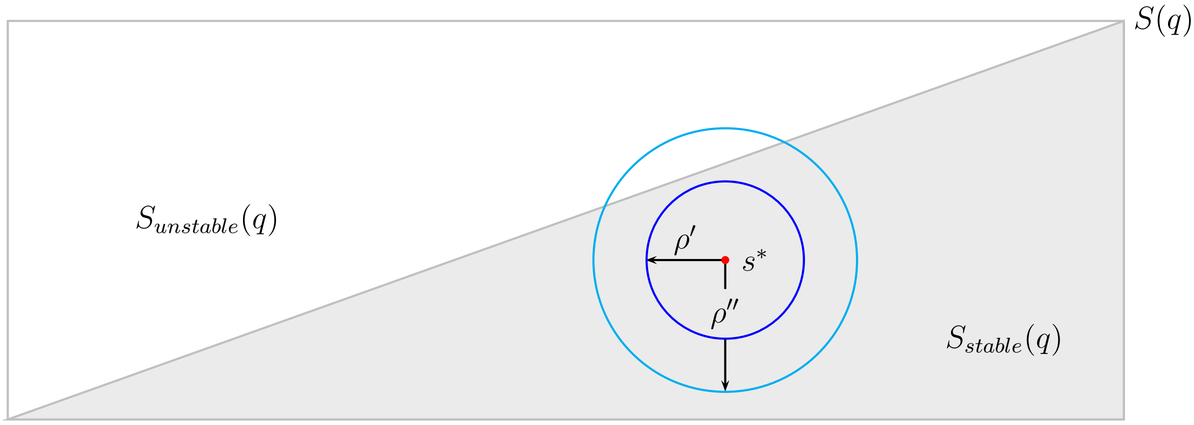


Figure 4.3: Local worst-case analysis of the *Radius of Stability* model

4.3.2 An info-gap decision theory perspective

Recall that info-gap's robustness model is a *Radius of Stability* model characterized by

$$S(q) = \mathscr{U} \tag{4.3}$$

$$S_{stable}(q) = \{s \in \mathscr{U} : r^* \leq r(q, s)\} \tag{4.4}$$

Thus, the above analysis applies here in full. This means that formally, info-gap's robustness model conducts a local worst-case analysis with respect to the constraint $r^* \leq r(q, u)$ on a ball $B(\rho, s^*)$.

Hence, for any given pair (q, ρ), there is at least one worst s in $B(\rho, s^*)$, meaning that the existence of a worst case is not an issue here.

The situation is similar with regard to info-gap's decision model: for any given pair (q, ρ) info-gap's decision model conducts a local worst-case analysis with respect to $r^* \leq r(q, u)$ on a ball $B(\rho, s^*)$. There is at least one worst s in $B(\rho, s^*)$ so the existence of a worst case is not an issue.

In short, info-gap's generic robustness model is a local worst-case model par excellence, where the radius ρ plays the role of the parameter y alluded to above.

Remark

It should be pointed out that the distinction between a worst-case analysis of a *constraint* and a worst-case analysis of a performance *level* is stylistic in the sense that the former can be reformulated as a worst-case analysis of a suitably constructed performance level.

To illustrate, in the framework of the *Radius of Stability* model, we can regard the local worst-case analysis associated with (4.2) as a local worst-case analysis on $B(\rho, s^*)$ associated with the performance function φ defined as follows:

$$\varphi(q, \rho, s) := \begin{cases} \rho & , \ s \in S_{stable}(q) \\ -\infty & , \ s \notin S_{stable}(q) \end{cases} , \ q \in Q, \rho \geq 0, s \in S(q) \tag{4.5}$$

and where "larger is better".

In this framework, for a given (q, ρ) pair, the worst s in $B(\rho, s^*)$ is any s in $B(\rho, s^*)$ that minimizes $\varphi(q, \rho, s)$ (with respect to s) on $B(\rho, s^*)$.

Note that $\varphi(q, \rho, s)$ can take only two possible values, namely ρ and $-\infty$. Hence, for any given pair (q, ρ), there is at least one worst s in $B(\rho, s^*)$, meaning that the existence of a worst case is not an issue here. This worst-case analysis is equivalent to a worst-case analysis of the constraint $s \in S_{stable}(q)$.

Similarly, in the case of info-gap decision theory, we can let

$$\varphi(q, \rho, s) := \begin{cases} \rho & , \ r^* \leq r(q, s) \\ -\infty & , \ r^* > r(q, s) \end{cases} \tag{4.6}$$

and assume that "larger is better".

Note that the worst case analysis of $s \in B(\rho, s^*)$ with respect to $r^* \leq r(q, s)$ is equivalent to the the worst case analysis of $s \in B(\rho, s^*)$ with respect to $\varphi(q, \rho, s)$.

4.4 Discussion

By definition then, *Radius of Stability* models, such as info-gap's robustness model, conduct their local worst-case analysis on a ball $B(\rho, s^*)$, where the radius of the ball (ρ) is specified by the decision maker, who aims to obtain the largest radius possible — subject to a performance constraint. The robustness analysis is conducted for one ball (value of ρ) at a time.

This being the case, the question obviously arising is:

> How can it possibly be claimed that info-gap's robustness analysis is not a worst case analysis?

A careful examination of the info-gap literature (in particular Ben-Haim 2001, 2006, 2007, 2010) reveals that such claims are based on a lack of awareness of the distinction between *local* and *global* worst-case analysis and a lack of appreciation of the difference between a worst-case analysis of a *performance* **level** and a worst-case analysis of a *performance* **constraint**.

More specifically.

Consider info-gap's generic robustness model

$$\hat{\alpha}(q) := \max \{\alpha \geq 0 : r^* \leq r(q, u), \forall u \in U(\alpha, \tilde{u})\} \ , \ q \in Q \tag{4.7}$$

It is patently clear that for any given (q, α) pair

- The analysis of the admissibility of this pair is conducted on $U(\alpha, \tilde{u})$, not on $\mathscr{U}$.
- The worst case of u sought for this pair is not with respect to $r(q, u)$, but rather with respect to $r^* \leq r(q, u)$.

As for the latter, note that the worst $u \in U(\alpha, \tilde{u})$ with respect to $r^* \leq r(q,u)$ is not necessarily the worst $u \in U(\alpha, \tilde{u})$ with respect to $r(q,u)$. For instance, it can be any $u \in U(\alpha, \tilde{u})$ such that $r^* > r(q,u)$, not necessarily a $u \in U(\alpha, \tilde{u})$ that minimizes $r(q,u)$ over $u \in U(\alpha, \tilde{u})$.

But, the argument in the info-gap literature to explain why info-gap's robustness analysis is not a worst-case analysis essentially runs as follows:

1. Clearly, an examination of info-gap's robustness model reveals that info-gap's robustness analysis is not a worst-case analysis with respect to $r(q,u)$ over the entire uncertainty space $\mathscr{U}$.
2. That is, info-gap's robustness analysis does not determine the robustness of decision q on the basis of the worst u in $\mathscr{U}$ with respect to the performance level $r(q,u)$.
3. Indeed, to begin with, in general (e.g. when $\mathscr{U}$ is **unbounded**), there might not even be a worst u in $\mathscr{U}$ with respect to $r(q,u)$.

And so, on this argument, info-gap's robustness analysis is definitely not a worst-case analysis.

But as can be gathered from the preceding discussion, this argument is a non-argument, because it seeks to make a case for the wrong point.

For the argument to have any merit at all, it must be based on what info-gap's robustness analysis does, and not on what info-gap's robustness analysis does not do. The crucial point is that, **by definition,** info-gap's robustness analysis is conducted **locally** on $U(\alpha, \tilde{u})$ rather than globally on $\mathscr{U}$, and what is more, that this analysis is carried out with respect to the constraint $r^* \leq r(q,u)$, rather then performanace level $r(q,u)$. **This is what info-gap's robustness analysis does**. So, it is only on grounds of this specific analysis that one can possibly argue one way or the other.

The facts are then as follows:

(i) Clearly, for any given (q, α) pair, info-gap's robustness analysis **is not a global** worst-case analysis with respect to the **performance level** $r(q,u)$ over the entire uncertainty space $\mathscr{U}$.

(ii) Clearly, for any given (q, α) pair, info-gap's robustness analysis **is a local** worst-case analysis with respect to the **constraint** $r^* \leq r(q,u)$ over the ball $U(\alpha, \tilde{u})$.

The role of the amplifier "clearly" in (i) is to emphasize that in (4.7) the object of interest is not $r(q,u)$ as such but rather $r^* \leq r(q,u)$ and that this requirement is imposed on values of u in $U(\alpha, \tilde{u})$ not in $\mathscr{U}$.

And the role of the amplifier "clearly" in (ii) is to emphasize that in (4.7), the object of interest is represented by $r^* \leq r(q,u)$ and that the worst u in $U(\alpha, \tilde{u})$ is therefore any element of $U(\alpha, \tilde{u})$ that yields the worst result for $r^* \leq r(q,u)$ over $u \in U(\alpha, \tilde{u})$, observing that there are at most two possible results, namely "satisfied" and "violated".

Remark:
It is interesting to note that for all the claims vehemently denying that info-gap's robustness

analysis is a worst-case analysis, there are claims in some info-gap publications arguing the precise opposite. To illustrate this contradiction, consider again the following (emphasis added):

> **Info-gap theory is not a worst-case analysis.** While there may be a worst case, one cannot know what it is and one should not base one's policy upon guesses of what it might be. Info-gap theory is related to robust-control and min-max methods, but nonetheless different from them. The strategy advocated here is not the amelioration of purportedly worst cases.
>
> Ben-Haim (2010, p. 9)

And compare it with this (emphasis added):

> For the application, the optimization searches for the model that yields the **worst possible** test-analysis correlation metric $R(q; u)$ **at each uncertainty level.**
>
> Hemez, Doebling, and Ben-Haim (2003, p. 10)

> On the left of Eq. (13), searching for **the worst** test-analysis correlation error **at each uncertainty level** provides the robustness function α.
>
> Hemez and Ben-Haim (2004, pp. 1458-9)

The caption of Fig. 7 in Hemez and Ben-Haim (2004) reads as follows (emphasis added):

> Results of the **worst-case info-gap robustness analysis**.

Similar views are expressed in Hemez and Ben-Haim (2002).

4.5 Summary

The moral of the story is that classic worst-case analysis paradigms, such as *Wald's Maximin model* (see Chapter 5), allow the analyst to determine the scope of the worst-case analysis. Therefore, it is extremely important to distinguish between **local** and **global** worst-case analyses.

The implications for info-gap decision theory are clear. For all the emphatic statements in the info-gap literature denying this fact, info-gap's robustness analysis is a (local) worst-case analysis.

It is important to take full note of this fact because of the huge knowledge base that is available on worst-case analysis. Unfounded claims in the info-gap literature by senior info-gap scholars that info-gap robustness analysis is not a worst-case analysis may unwittingly deprive users of info-gap decision theory of this important, relevant resource.

Chapter 5

Wald's Maximin model

5.1 Introduction

Before I get down to the technical discussion of this important topic, I want to explain my decision to incorporate this topic in the main body of the document, rather than place it in an appendix, where it perhaps belongs. Indeed, in earlier drafts of this document this discussion was included as an appendix.

The point is that given the central role of Wald's Maximin model in classical decision theory and robust decision-making, one would have assumed that info-gap scholars are conversant with this topic. This suggests that this topic should be regarded as "background material" whose proper place is in an appendix. Furthermore, given the reference in Bersesford-Smith and Thompson (2009) to the formal proof in Sniedovich (2007) that info-gap's robustness model is a Maximin model, I assumed that this important message finally got across to be accepted for what it is . . . a formal proof that info-gap's robustness model is a Maximin model.

However, in view of the following, and similar statements in other recent articles, I changed my mind and decided to relocate this topic to the main body of the document. For, as you can clearly see, senior info-gap scholars, including the Father of info-gap decision theory, continue to insist that info-gap's robustness model is not a Maximin model (emphasis added)[1]:

> Various suggestions have been proposed for making decisions in competitive strategic games. A common one is what is called the "minimax" strategy: choose the option with which you do as well as you can if the worst happens. Though you can't specify how likely it is that the worst will happen, adopting this strategy is a kind of insurance policy against total disaster. **Minimax is a kind of cousin to robust satisficing, but it is not the same.** First, at least sometimes, you can't even specify what the worst possible outcome can bring. In such situations, a minimax strategy is unhelpful. Second, and more important, robust satisficing is a way to manage uncertainty, not a way to manage bad outcomes. In choosing Brown over Swarthmore, you are not insuring a tolerable outcome if the worst happens. You are acting to produce a good-enough outcome if any of a large number of things happen. There are certainly situations in which minimax

[1] For the record I should point out that Prof. Ben-Haim is aware of the existence of formal proofs that info-gap's robustness model is a Maximin model.

> strategies make sense.
>
> Barry Schwartz, Yakov Ben-Haim, and Cliff Dacso (2011, p. 25)

So, one of the objectives of the discussion in this chapter is to clarify, yet again, the family ties between Wald's Maximin model and info-gap's robustness model. As we shall see, the former is the Grandmother of the latter, the *Radius of Stability model* being its Mother.

Indeed, that such a clarification is vital is born out by the fact that erroneous unfounded statements about Wald's Maximin model — statements that have the potential to mislead scholars/analysts who are not at home with this topic — are continued to be bandied about carelessly in the info-gap literature. For instance, consider this statement (emphasis added):

> In a sense info gap analysis may be thought of as extended and structured sensitivity analysis of preference orderings between options. **While there is a superficial similarity with minimax decision making,** no fixed bounds are imposed on the set of possibilities, leading to a comprehensive search of the set of possibilities and construction of functions that describe the results of that search.
>
> Hine and Hall (2010, pp. 16-17)

And in view of statements such as the following, it is vital to revisit the question: what kind of worst-case analysis is conducted by the Maximin model?

> **Relation to the Min-Max Strategy.** The min-max strategy selects the design that minimizes the maximal loss. The infogap robustness function has a formal relation to the min-max strategy. However, there are two important differences. First, implementation of a min-max strategy requires knowledge of a worst case. In contrast, an info-gap model of uncertainty is explicitly designed to represent situations in which we do not know how wrong the best estimate can be. Second, even if we reliably know the worst that can occur, we may not want to design for that contingency. The clearest case is when the outcome anticipated from the min-max design is unacceptable because it violates the performance requirements.
>
> Arkadeb Ghosal, Haibo Zeng, Marco Di Natale, Yakov Ben-Haim (2010, p. 2)

And in view of the following, it is necessary to point out that based as it is on an instance of Wald's Maximin model, info-gap decision theory cannot generalize Wald's Maximin strategy (emphasis added):

> Info-gap **generalizes** the maximin strategy by identifying worst-case outcomes at increasing levels (horizons) of uncertainty. This **permits** the construction of 'robustness curves' that describe the decay in guaranteed minimum performance (or worst-case outcome) as uncertainty increases.
>
> Wintle, Bekessy, Keith, van Wilgen, Cabeza, Schroder, Carvalho, Falcucci, Maiorano, Regan, Rondinini, Boitani and Possingham (2011, p. 357)

See Appendix J for more details.

With this in mind, let us now examine this stalwart of decision theory and robust optimization.

5.2 Wald's Maximin rule

Wald's Maximin model is the foremost tool used in worst-case analysis modeling, hence its prominence in decision-making under severe uncertainty and robust optimization.

It is important to note though that, much as this versatile model dominates the scene in decision-making under severe uncertainty, neither conceptually nor technically does it require that the decision-making situations concerned be subject to *uncertainty.* It provides a means to evaluate and rank decisions on the basis of their *worst-case performance.*

According to this paradigm then, the best decision is that whose worst-case performance is at least as good as the worst-case performance of the other decisions under consideration. Hence[2],

Maximin Rule
Rank alternatives/decisions on the basis of their worst-case performance. Hence, select an alternative/decision whose worst-case performance is at least as good as the worst case performance of all other alternatives/decisions.

Note that in the context of decision-making under uncertainty, *Wald's Maximin model* trades "uncertain outcomes" for "certain bleak outcomes". Or in other words, it trades the convenience of a certain, but "grim world", for the inconvenience of a severely uncertain but "mixed world". Needless to say, the hope is that the worst case will not be realized: *hope for the best but plan for the worst!*

This is illustrated in Table 5.1, where the "certain" worst-case states are shown for each alternative.

		Nature					
		s_1	s_2	s_3	s_4	s_5	S^*
DM	a_1	3	$\boxed{2}$	5	6	$\boxed{2}$	$\{s_2, s_5\}$
	a_2	9	8	$\boxed{1}$	6	7	$\{s_3\}$
	a_3	$\boxed{0}$	5	4	3	$\boxed{0}$	$\{s_1, s_5\}$

Table 5.1: Expected Payoffs, S^* = set of worst-case states

Insofar as "robustness" is concerned, as in classical *game theory*, *classical decision theory* employs the notion SECURITY LEVEL to measure the performance of an alternative against severe uncertainty. Formally, the *security level of an alternative* is the payoff yielded by the worst-case state(s) pertaining to this alternative. Thus, for an $m \times n$ payoff table, the security level of action a_i is defined as follows:

$$SL(i) := \min_{1 \leq j \leq n} \textit{payoff}(i, j) \ , \ i = 1, 2, \ldots, m \tag{5.1}$$

[2]This is an adaptation of Rawls's (2005) formulation of the rule.

The optimal decision is therefore obtained by solving this problem:

$$z^* := \max_{1\leq i\leq m} SL(i) \tag{5.2}$$

$$= \max_{1\leq i\leq m} \min_{1\leq j\leq n} \textit{payoff}(i,j) \tag{5.3}$$

Table 5.2 illustrates this recipe in action.

		Nature						
		s_1	s_2	s_3	s_4	s_5	SL	S^*
DM	a_1	3	[2]	5	6	[2]	[[2]]	$\{s_2, s_5\}$
	a_2	9	8	[1]	6	7	1	$\{s_3\}$
	a_3	[0]	5	4	3	[0]	0	$\{s_1, s_5\}$

Table 5.2: Wald's Maximin analysis

The highest security level — equal to 2 — is associated with a_1. Hence, according to the *Maximin rule,* a_1 is the best (most robust) alternative available to the DM. If the DM selects this alternative, then regardless of what state will be realized, her payoff will not be less than 2. There are two worst-case states for this alternatives, namely $S^*(1) = \{s_2, s_5\}$.

The main drawback of this model is that it can yield extremely "conservative" results. For example, consider the payoff table shown in Table 5.3.

		Nature						
		s_1	s_2	s_3	s_4	s_5	SL	S^*
DM	a_1	3	[2]	5	6	[2]	2	$\{s_2, s_5\}$
	a_2	900	800	[1]	600	700	1	$\{s_3\}$
	a_3	[3]	5	4	6	[3]	[[3]]	$\{s_1, s_5\}$

Table 5.3: Wald's Maximin analysis

According to the *Maximin rule,* alternative a_2 is the least attractive, even though four out of its five possible payoffs are much higher than those associated with the other two alternatives. In a word, [one bad apple] tells against the performance of an otherwise highly attractive alternative.

Another feature that, to the minds of some experts, renders this paradigm "problematic" is that:

> An addition of a **constant** to the payoffs associated with a given state (column of the payoff table) may change the ranking of the alternatives.

This is illustrated in Table 5.4 which is obtained by adding 3 to the payoffs associated with state s_3.

Note that in the case of the payoffs listed in Table 5.3, alternative a_2 is the least attractive, whereas according to those listed in Table 5.4 this alternative is the most attractive.

		Nature					
		s_1	s_2	s_3	s_4	s_5	SL	S^*
DM	a_1	3	[2]	8	6	[2]	2	$\{s_2, s_5\}$
	a_2	900	800	[4]	600	700	[[4]]	$\{s_3\}$
	a_3	[3]	5	7	6	[3]	3	$\{s_1, s_5\}$

Table 5.4: Wald's Maximin analysis

So, the fault that some experts find with this paradigm is that an addition of the same constant to all the payoffs associated with a given state (column) should not have the drastic effect of altering the alternatives' ranking.

It should be pointed out that this problem does not affect the decision rule based on *Laplace's Principle of Insufficient Reason* (see Appendix C). This decision rule has, however, a fault of its own which is that its ranking of alternatives may change if a column of the payoff table is **duplicated**. This difficulty does not affect the rule stipulated by *Wald's Maximin paradigm.*

5.2.1 Variations on a theme

I refer the reader to Resnik (1987) and French (1988) for details on the variants of *Wald's Maximin model.* Their common denominator is that they end converting the *severe uncertainty* into *certainty* by considering the worst/best case payoffs (or regrets) associated with the states.

For the benefit of info-gap scholars, it is instructive to illustrate this point in the context of the Minimin model given that *info-gap's opportuneness model* is in fact a simple instance of the classic *Minimin model.*

So, recall that in the framework of this model, Nature cooperates with the DM, meaning that She selects the best-case state pertaining to the alternative selected by the DM. The formal model is then as follows:

$$z^\circ := \min_{1\leq i\leq m} \min_{1\leq j\leq n} \textit{payoff}(i,j) \tag{5.4}$$

where the outer $\min$ represents the DM and the inner $\min$ represents Nature.

Because Nature cooperates with the DM, the decision-making situation represented by this model can be viewed as a simple optimization problem, involving only the decision maker, who selects both the alternative (a_i) and the state (s_j) with the view to *minimize her payoff.* Hence, symbolically,

$$z^\circ := \min_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \textit{payoff}(i,j) \tag{5.5}$$

This ultra-optimistic paradigm forms part of *Hurwicz's model,* namely the *Optimism Pessimism Index,* where its function is to moderate the extreme pessimism of *Wald's Maximin model* (Resnik 1987 and French 1988).

I need hardly point out, though, that classical decision theory does not propose that the ultra-optimistic *Minimin paradigm* be used on it own, as one would be hard pressed to justify the logic behind the employment of such a model in practice.

5.3 Maximin game

It is instructive to describe the *Maximin model* as a game between two players: The decision Maker (DM) and Nature. The former controls the *decision variable*, the latter the *state variable*. DM plays first, aiming to *maximize* her reward, whereupon Nature responds, aiming to *minimize* the payoff awarded to DM.

Here is a more formal description of the game:

Maximin Game

- Step 1: DM selects a decision $x \in X$.
- Step 2: Nature selects the worst state in $S(x)$, call it $s(x)$.
- Step 3: A payoff $f(x, s(x))$ is awarded to DM.

We refer to X as the *decision space*, to $S(x)$ as the *state space* associated with decision x and to f as the *reward/payoff function*. Note that when Nature selects her state in $S(x)$, she knows what decision was selected by DM. This conceptual model is shown in Figure 5.1.

$$\overset{DM}{x \in X} \longrightarrow \overset{Nature}{s(x) \in S(x)} \longrightarrow \overset{Payoff}{f(x, s(x))}$$

Figure 5.1: Maximin game

The mathematical formulation of this game is as follows[3]:

$$p^* := \max_{x \in X} \min_{s \in S(x)} f(x, s) \tag{5.6}$$

observing that the outer $\max$ represents DM and the inner $\min$ represents Nature.

5.4 Mathematical programming format

Often, it is more convenient to express the *Maximin model* as a model of a standard maximization problem, that is a problem of the following generic form

$$\max_{y \in Y} g(y) \text{ subject to some constraints on y} \tag{5.7}$$

[3]We assume here that the problem is nice and smooth so that the $\max$ and $\min$ exist.

The following observation can be useful in such cases:

$$\underset{\text{Classic format}}{\max_{x\in X}\min_{s\in S(x)} f(x,s)} \quad \equiv \quad \underset{\text{MP format}}{\max_{x\in X, v\in\mathbb{R}}\{v : v \leq f(x,s), \forall s\in S(x)\}} \tag{5.8}$$

Note that if (x^*, v^*) is an optimal solution to the problem specified by the RHS of (5.8), then

$$v^* = f(x^*, v^*) = \min_{s\in S(x^*)} f(x^*, s) \tag{5.9}$$

The transformation from the Classic format to the MP (mathematical programming) format is obtained by means of an appeal to the fact that

$$\min_{y\in Y} g(y) \equiv \max_{v\in\mathbb{R}} \{v : v \leq g(y), \forall y \in Y\} \tag{5.10}$$

given that the $\min$ is attained. This “trick” is used extensively in game theory, optimization theory, robust optimization, and so on.

5.5 Constrained Maximin models

One of the advantages of the MP format is that it can easily incorporate *constraints* in the formulation of a Maximin model. For example, suppose that we want to incorporate the constraint

$$g(x,s) \in G(x)\ ,\ \forall s \in S(x) \tag{5.11}$$

in the *Maximin model* specified by (5.8).

In the framework of the MP format, this task is straightforward. To wit, the new (adjusted) *Maximin model* is as follows

$$\max_{x\in X, v\in\mathbb{R}}\{v : v \leq f(x,s), g(x,s) \in G(x), \forall s \in S(x)\} \tag{5.12}$$

On the other hand, the equivalent Classic format would be

$$\max_{x\in X}\min_{s\in S(x)} h(x,s) \tag{5.13}$$

where

$$h(x,s) = \begin{cases} f(x,s) & , \ g(x,s) \in G(x) \\ -\infty & , \ g(x,s) \notin G(x) \end{cases} \ ,\ x \in X, s \in S(x) \tag{5.14}$$

The large penalty ($-\infty$) in (5.14) for violating the constraint $g(x,s) \in G(x)$ is intended to deter the DM from selecting an $x \in X$ that violates this constraint for some $s \in S(x)$. This, in turn will stop Nature from selecting a state $s \in S(x)$ that violates this constraint for the decision selected by DM.

It goes without saying that this constraint can be incorporated explicitly in the Classic format as follows:

$$\max_{x\in X}\min_{s\in S(x)}\{f(x,s):g(s)\in G(x),\forall s\in S(x)\} \tag{5.15}$$

This is a perfectly kosher *Maximin model* and so are

$$\max_{x\in X}\min_{s\in S(x)}\{f(x,s):s\in G(x),\forall s\in S(x)\} \tag{5.16}$$

and

$$\max_{x\in X}\min_{s\in S(x)}\{f(x,s):r^{\circ}\leq R(q,s),\forall s\in S(x)\} \tag{5.17}$$

where r° is a given numerical scalar and R is a real-valued function of the decision and state variables.

Note that if $f(x,s)$ is independent of s, namely if robustness is not sought with respect to the objective function f, then these two models will take the following much simpler forms, respectively:

$$\max_{x\in X}\min_{s\in S(x)}\{f(x):s\in G(x),\forall s\in S(x)\}\equiv\max_{x\in X}\{f(x):s\in G(q),\forall s\in S(x)\} \tag{5.18}$$

$$\max_{x\in X}\min_{s\in S(x)}\{f(x):r^{\circ}\leq R(q,s),\forall s\in S(x)\}\equiv\max_{x\in X}\{f(x):r^{*}\leq r(q,s),\forall s\in S(x)\} \tag{5.19}$$

Also note that the simplified versions on the right hand sides of (5.18) and (5.19) — manifested in the absence of the iconic $\min\limits_{s\in S(x)}$ of the Classic format — give expression to the fact that these Maximin models seek robustness only with respect to the constraints, not with respect to the objective function f.

All this goes to show that the *Maximin paradigm* puts at the analyst's disposal a highly pliable, hence versatile modeling framework, enabling the choice between the Classic format, the MP format, and other formats such as (5.15)-(5.19).

5.6 Relation to the Radius of Stability model

Consider the generic *Radius of Stability* model, namely

$$\rho(q,\tilde{s}):=\max_{\rho\geq 0}\{\rho:s\in S_{stable}(q),\forall s\in B(\rho,\tilde{s})\}\ ,\ q\in Q \tag{5.20}$$

Theorem 5.6.1 *The generic Radius of Stability model is an instance of Wald's generic Maximin model.*

Proof. To show that (5.20) is an instance (special case) of Wald's generic Maximin model, we show that (5.20) is an instance of the model specified by the right hand side of (5.18).

Consider then the instance of specified by

$$x = \rho \quad (5.21)$$

$$X = [0, \infty) \quad (5.22)$$

$$S(x) = B(\rho, \tilde{s}) \ , \ x = \rho \in X \quad (5.23)$$

$$f(x) = \rho \ , \ x = \rho \in X \quad (5.24)$$

$$G(x) = S_{stable}(q) \ , x = \rho \in X \quad (5.25)$$

where q is given. In this case we would have

$$\overbrace{\max_{x \in X} \{f(x) : s \in G(x), \forall s \in S(x)\}}^{\text{Maximin model}} \equiv \max_{x \geq 0} \{x : s \in S_{stable}(q), \forall s \in B(x, \tilde{s})\} \quad (5.26)$$

$$\equiv \overbrace{\max_{\rho \geq 0} \{\rho : s \in S_{stable}(q), \forall s \in B(\rho, \tilde{s})\}}^{\text{Radius of Stability model}} \quad (5.27)$$

We therefore conclude that the generic *Radius of Stability model* specified above is equivalent to the Maximin models specified in (5.18). This implies that the *Radius of Stability* model specified in (5.20) is an instance of Wald's generic *Maximin model*. QED

The $\equiv$ sign indicates that these models are *equivalent.* That is, not only are the optimal values of the objective functions the same, the optimal solutions (decision variables) are also the same, and there is a clear correspondence between their constituent constructs, as specified by (5.22)-(5.25).

Since we have already established that *info-gap's robustness model* is a *Radius of Stability* model, we conclude that:

Theorem 5.6.2 *Info-gap's robustness model is an instance of Wald's generic Maximin model.*

However, as a modeling exercise, let us prove this result explicitly. So recall that info-gap's generic robustness model is as follows:

$$\max_{\alpha \geq 0} \{\alpha : r^* \leq r(q, u), \forall u \in U(\alpha, \tilde{u})\} \ , \ q \in Q \quad (5.28)$$

Proof. To show that info-gap's generic robustness model is an instance of Wald's generic Maximin model, consider the Maximin model specified by the right hand side of (5.19), namely

$$\max_{x \in X} \{f(x) : r^{\circ} \leq R(x, s), \forall s \in S(x)\} \quad (5.29)$$

Now let

$$x = \alpha \quad (5.30)$$

$$s = u \quad (5.31)$$

$$X = [0, \infty) \quad (5.32)$$

$$S(x) = U(\alpha, \tilde{u}) \ , \ x \in X \quad (5.33)$$

$$f(x) = \alpha\ ,\ x = \alpha \in X \tag{5.34}$$

$$R(x,s) = r(q,u), x = \alpha \in X, s = u \in S(x)\ , \quad (q \in Q \text{ is given}) \tag{5.35}$$

$$r^\circ = r^* \tag{5.36}$$

In this case we would have

$$\max_{x\in X} \{f(x) : r^\circ \leq R(x,s), \forall s \in S(x)\} \equiv \max_{\alpha\geq 0} \{\alpha : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u}\} \tag{5.37}$$

We therefore conclude that info-gap's robustness model is equivalent to the Maximin model specified by (5.29). The implication is therefore that info-gap's robustness model is an instance of Wald's generic *Maximin model*. QED

I note again that the simplification due to the absence of the iconic $\min_{s\in S(x)}$ in this Maximin model is a manifestation of the fact that this Maximin model seeks robustness only with respect to the constraints, not with respect to the objective function.

5.7 Relation to info-gap's decision model

Info-gap decision theory ranks decisions on the basis of their robustness: the more robust the better. Hence, the optimal decision is one whose robustness is the largest. Consequently, in view of Theorem 5.6.2, we conclude:

Theorem 5.7.1 *Info-gap's generic decision model is an instance of Wald's generic Maximin model.*

Proof. This is obtained directly from Theorem 5.6.2 by incorporating $q \in Q$ as a decision variable in the models. In other words, info-gap's decision model, namely

$$\max_{q\in Q} \max_{\alpha\geq 0} \{\alpha : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{5.38}$$

is the instance of the maximin model

$$\max_{x\in X} \min_{s\in S(x)} \{f(x,s) : r^* \leq R(x,s), \forall s \in S(x)\} \tag{5.39}$$

specified by

$$x = (q,\alpha) \tag{5.40}$$

$$s = u \tag{5.41}$$

$$X = Q \times [0,\infty) \tag{5.42}$$

$$S(x) = B(\alpha,\tilde{u}),\ x = (q,\alpha) \in X \tag{5.43}$$

$$f(x,s) = \alpha,\ x = (q,\alpha) \in X, u = s \in S(x) \tag{5.44}$$

$$R(x,s) = r(q,u), x = (q,\alpha) \in X, u = s \in S(x) \tag{5.45}$$

$$r^\circ = r^* \tag{5.46}$$

To ascertain that this is indeed so, observe that since $f(x,u) = f((q,\alpha),u) = \alpha$ is independent of u, we have

$$\max_{x\in X}\min_{s\in S(x)} \{f(x,s) : r^{\circ} \le R(x,s), \forall s \in S(x)\} \tag{5.47}$$

$$\equiv \max_{q\in Q,\alpha\ge 0}\min_{u\in U(\alpha,\tilde{u})} \{\alpha : r^{*} \le r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{5.48}$$

$$\equiv \max_{q\in Q,\alpha\ge 0} \{\alpha : r^{*} \le r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{5.49}$$

$$\equiv \max_{q\in Q}\max_{\alpha\ge 0} \{\alpha : r^{*} \le r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{5.50}$$

QED

There are, of course, other Maximin representations of info-gap's robustness and info-gap's models. For example, if we let

$$\varphi(q,\alpha,u) := \begin{cases} \alpha & , \ r^{*} \le r(q,u) \\ -\alpha & , \ r^{*} > r(q,u) \end{cases} \ , \ q \in Q, \alpha \ge 0, u \in U(\alpha,\tilde{u}) \tag{5.51}$$

then

$$\max_{\alpha\ge 0} \{\alpha : r^{*} \le r(q,u), \forall u \in U(\alpha,\tilde{u})\} \equiv \max_{\alpha\ge 0}\min_{u\in U(\alpha,\tilde{u})} \varphi(q,\alpha,u) \ , \ q \in Q \tag{5.52}$$

The comment above regarding the local nature of the worst-case analysis conducted by this model applies here as well.

5.8 Role in robust decision-making

The centrality of Wald's Maximin model in robust decision-making is attested by its prominence in the robust optimization literature. Thus, a quick scan of this literature immediately reveals that this versatile model — and its many variants — in fact dominate the scene in this discipline. But not only there.

Consider for example the following three quotes. The first is the abstract of the entry *Robust Control* by Noah Williams (2008) in the *New Palgrave Dictionary of Economics*:

> Robust control is an approach for confronting model uncertainty in decision making, aiming at finding decision rules which perform well across a range of alternative models. This typically leads to a minimax approach, where the robust decision rule minimizes the worst-case outcome from the possible set. This article discusses the rationale for robust decisions, the background literature in control theory, and different approaches which have been used in economics, including the most prominent approach due to Hansen and Sargent.

The second is from the book *Robust Statistics* (Huber and Ronchetti, 2009, p. 17):

> But as we defined robustness to mean insensitivity with regard to small deviations from assumptions, any quantitative measure of robustness must somehow be concerned with the maximum degradation of performance possible for an ϵ-deviation from the assumptions. The optimally robust procedure minimizes this degradation and hence will be a minimax procedure of some kind.

The third is from the paper *Solution of macro-models with Hansen-Sargent robust policies: some extensions* by Paolo Giordani and Paul Söderlind (2004, p. 2370, emphasis added):

> **From a technical point of view, robustness involves a switch from a minimization problem (minimizing a loss function) to an appropriately specified min-max problem.** In order to set up and solve a min-max problem, it is convenient to work with a two-agent representation: the policy function selected by the planner is the equilibrium outcome of a two person game in which a fictitious evil agent, whose only goal is to maximize the planner's loss, chooses a model from the available set, and the planner chooses a policy function.

In short, given the high profile that *Wald's Maximin model* and its numerous variates and special cases have in an array of areas concerned with robust decision-making, it is hard to see how a theory claiming to offer a new method for robust decision-making is not carefully compared to these established paradigms.

5.9 The Size Criterion revisited

Recall that the *Size Criterion* (see section 2.3) is the most "intutive" measure of global robustness. In Appendix C it is shown that in cases where the uncertainty space is discrete, this criterion can be formulated as a rule governed by *Laplace's Principle of Insufficient Reason.* Here I show that it can also be formulated as a Maximin rule.

So consider the following simple Maximin model:

$$z(q) := \max_{V\subseteq\mathscr{U}} \min_{u\in V} f(V,u)\ ,\ f(V,u) := \begin{cases} size(V) & ,\ u \in \mathscr{U}(q) \\ -\infty & ,\ u \notin \mathscr{U}(q) \end{cases}\ ,\ q\in Q \tag{5.53}$$

where for each $q \in Q$, $\mathscr{U}(q)$ is a subset of $\mathscr{U}$. It follows then that

$$z(q) = \max_{V\subseteq\mathscr{U}} \min_{u\in V} f(V,u) \tag{5.54}$$

$$= \max_{V\subseteq\mathscr{U}} \{size(V) : u \in \mathscr{U}(q), \forall u \in V\} \tag{5.55}$$

$$= \max_{V\subseteq\mathscr{U}} \{size(V) : V \subseteq \mathscr{U}(q)\} \tag{5.56}$$

$$= size(\mathscr{U}(q)) \ \ \text{(note that } V \subseteq \mathscr{U}(q) \text{ entails that } size(V) \leq size(\mathscr{U}(q))\text{)} \tag{5.57}$$

In short, the above Maximin model is equivalent to the *Size Criterion.*

5.10 A robust optimization perspective

Analysts who are not conversant with Wald's Maximin model ought to take note that the three generic Maximin models formulated in this chapter (namely the Classic format, the MP format, and the constrained version) are in fact *equivalent*. This is so because, despite the slight differences in their respective mathematical formulations, all three represent the same basic model that gives expression to *Wald's Maximin Rule*.

It is instructive to compare these models from the standpoint of *robust optimization*, and for this purpose consider the following abstract (constrained) *optimization problem*:

$$\text{Problem P}:$$
$$z^* := \max_{x \in X} f(x) \;\text{ subject to }\; constraints(x) \tag{5.58}$$

where X is some set, f is a real valued function on X, and $constraints(x)$ denotes a set of constraints on the decision variable x.

Now, suppose that the objective function f and/or the constraints depend on some parameter $s \in \mathcal{S}$. Then we can distinguish between three families of *parametric problems* induced by Problem P, namely

$$\text{Problem P(s)}, s \in \mathcal{S}:$$
$$z^*(s) := \max_{x \in X} f(x;s) \;\text{ subject to }\; constraints(x;s) \tag{5.59}$$

$$\text{Problem F(s)}, s \in \mathcal{S}:$$
$$z^*(s) := \max_{x \in X} f(x;s) \tag{5.60}$$

$$\text{Problem C(s)}, s \in \mathcal{S}:$$
$$z^*(s) := \max_{x \in X} f(x) \;\text{ subject to }\; constraints(x;s) \tag{5.61}$$

where the "$x;s$" notation is used to distinguish between the decision variable x and the parameter s. For instance, $f(x;s)$ indicates that x is the "official" argument of f whereas s is "just" a parameter of this function.

It is important to note that the absence of explicitly stated constraints in the formulation of Problem F(s) simply means that the constraints, if there are any, are incorporated in the definition of set X and/or the definition of the objective function f.

The robustness issue that these parametric models address is as follows:

> Find a solution $x \in X$ that is **robust** against variations in the value of s over $\mathcal{S}$. That is, identify a solution $x \in X$ that performs "well" as s varies over $\mathcal{S}$.

If to determine how well x performs as s varies over $\mathcal{S}$, we apply the pessimistic *worst-case* approach adopted by the *Maximin Rule*, we obtain the following *robust-counterparts* of the above parametric models:

Maximin: Full Monty Model:

$$z^* := \max_{x\in X} \min_{s\in S(x)} \{f(x,s) : \mathit{constraints}(x;s), \forall s \in S(x)\} \tag{5.62}$$

Maximin: Decision Theory Model:

$$z^* := \max_{x\in X} \min_{s\in S(x)} f(x,s) \tag{5.63}$$

Maximin: Mathematical Programming Model:

$$z^* := \max_{x\in X} \{g(x) : \mathit{constraints}(x;s), \forall s \in S(x)\} \tag{5.64}$$

where $S(x)$ represents a subset of $\mathcal{S}$ associated with $x \in X$.

Note that in these models we regard f as a real valued function on $X \times \mathcal{S}$ and we use the objective function g, instead of f, in the formulation of the *Mathematical Programming Model*.

Also note that the *Full Monty Model* states explicitly that robustness is sought with respect to both the objective function f and the constraints. The others are its two obvious simplifications:

- The *Decision Theory Model* states that robustness is sought only with respect to the objective function f: there are apparently no explicit joint constraints on the *decision-state* pairs. This model is used extensively in introductory textbooks on *decision theory.*
- The *Mathematical Programming Model* states that robustness is sought only with respect to the constraints.

All the same, this does not imply that the *Full Monty Model* is more general than the other two models. Nor does it suggest that the *Mathematical Programming Model* cannot be applied in cases where robustness is sought with respect to the objective function.

In fact, it is easy to show that the *seemingly* far simpler *Mathematical Programming Model* is indeed as general as the *Full Monty Model*. Namely, that it "allows" the analyst to seek robustness with respect to an objective function as well as with respect to constraints.

To ascertain that this is indeed so, observe that by using the "$v \leq f(x,s)$ trick", the MP format of the *Full Monty Model* can be rewritten as follows:

$$z^* := \max_{x\in X} \min_{s\in S(x)} \{f(x,s) : \mathit{constraints}(x;s), \forall s \in S(x)\} \tag{5.65}$$

$$\equiv \max_{x\in X, v\in\mathbb{R}} \{v : v \leq f(x,s), \mathit{constraints}(x;s), \forall s \in S(x)\} \tag{5.66}$$

$$\equiv \max_{y\in Y} \{g(y) : \mathit{Constraints}(y;s), \forall s \in S(y)\} \tag{5.67}$$

where

$$y = (x,v) \tag{5.68}$$

$$Y = X \times \mathbb{R} \tag{5.69}$$

$$S(y) = S(x), y = (x,v) \in Y \tag{5.70}$$

Mathematical Programming Model of Wald's Maximin Rule

$$\max_{x\in X}\{g(x): constraints(x;s), \forall s\in S(x)\}$$

Instance	Instance
$x=\alpha$	$x=(q,\alpha)$
$s=u$	$s=u$
$X=[0,\infty)$	$X=Q\times[0,\infty)$
$S(x)=U(\alpha,\tilde{u})$	$S(x)=U(\alpha,\tilde{u})$
$g(x)=\alpha$	$g(x)=\alpha$
$constraints(x;s)=\{r(q,u)\le r^*\}$	$constraints(x;s)=\{r(q,u)\le r^*\}$
$\Downarrow$	$\Downarrow$
info-gap's robustness model	info-gap's decision model
$\max_{\alpha\ge 0}\{\alpha: r(q,u)\le r^*, \forall u\in U(\alpha,\tilde{u})\}$	$\max_{q\in Q}\max_{\alpha\ge 0}\{\alpha: r(q,u)\le r^*, \forall u\in U(\alpha,\tilde{u})\}$

Figure 5.2: A robust optimization perspective on info-gap models

$$g(y)=v, y=(x,v)\in Y \tag{5.71}$$

$$Constraints(y;s)=\{v\le f(x,s)\}\cup constraints(x;s), y=(x,v)\in Y, s\in S(y) \tag{5.72}$$

This means that the *Mathematical Programming model* (5.67) is equivalent to the *Full Monty model*. This, no doubt, explains why the *Mathematical Programming model* is used so widely in the *robust optimization* literature, even in situations where robustness is sought with respect to both the objective function and the constraints.

As might have been expected, this is also the generic Maximin model that most obviously subsumes info-gap's robustness model and info-gap's decision models as special cases (instances). This is shown in Figure 5.2.

So from a *robust optimization* point of view, info-gap's robustness model and info-gap's decision models are **simple** Maximin models in that

- There is only one (joint) constraint on the (decision,state) pairs. That is, the set $constraints(x;s)$ consists of a single (joint) constraint.
- Robustness is sought only with respect to this constraint — not with respect to the objective function.
- The objective function is linear with the decision variable (x) and does not depend on the state variable (s).

And the important lesson of this modeling exercise is that even if a mathematical model does not exhibit the iconic $\max\limits_{x\in X}\min\limits_{s\in S(x)}$ operation, **it does not follow that** the expression does not constitute a Maximin model. For example, the expression

$$\max_{y\in Y}\{h(y): 34\le g(y,t)\le 98, \forall t\in A(y)\} \tag{5.73}$$

represents a perfectly kosher Maximin model.

The iconic $\max_{y\in Y}$ and the fact that the objective function h does not depend on t, indicate that robustness (against variations in t) is not sought with respect to the objective function. The iconic $\forall t$, and the fact that the g depends on t, indicate that robustness is sought with respect to the constraint $34 \leq g(y,t) \leq 98$.

The reader may want to try his/her hand at establishing why the following two models,

$$\max_{\alpha\geq 0} \left\{\alpha : r^* \leq \min_{u\in U(\alpha,\tilde{u})} r(q,u)\right\}, q \in Q \tag{5.74}$$

and

$$\max_{\alpha\geq 0} \left\{\alpha : r^* \geq \max_{u\in U(\alpha,\tilde{u})} r(q,u)\right\}, q \in Q \tag{5.75}$$

are Maximin models, whereas these slightly different models, namely

$$\max_{\alpha\geq 0} \left\{\alpha : r^* \geq \min_{u\in U(\alpha,\tilde{u})} r(q,u)\right\}, q \in Q \tag{5.76}$$

and

$$\max_{\alpha\geq 0} \left\{\alpha : r^* \leq \max_{u\in U(\alpha,\tilde{u})} r(q,u)\right\}, q \in Q \tag{5.77}$$

are Maximax models (not Maximin models).

5.11 Bibliographic notes

It is most interesting that in the article in which he first described the generic Maximin model (actually Minimax model), Wald (1939) seemed not to be alive to the fact that the model was a ...Minimax model. Indeed, no reference is made in the paper to *game theory* nor to von Neumann's work.

The Minimax approach is outlined in the following short paragraph (Wald 1939, p. 305, emphasis added):

> There exist in general many system M_s which are admissible relative to the weight function given. The question arises as to how can we distinguish among them. Denote by r_{M_s} the **blue maximum** of the risk function corresponding to the system M_s of regions and to the given weight function. If we do not take into consideration a priori probabilities of $theta$, then it seems reasonable to choose that system M_s for which r_{M_s} becomes a **minimum.** We shall see in section 8 that the system M_s for which r_{M_s} becomes a minimum has some important properties which justify the distinction of this particular system of regions among all admissible systems.

The reference to *game theory* is made in Wald's (1945) famous paper entitled *Statistical decision functions which minimize the maximum risk*. The Minimax argument is as follows:

> However, in most of the applications not even the existence of such a priori probability distribution of θ can be postulated, and in those few cases where the existence of an a priori distribution of θ may be assumed this distribution is usually unknown. Under such circumstances it seems of interest to consider a decision function which minimizes the maximum (instead of some weighted average) of the risk function.
>
> Wald (1945, p. 267)

The relationship to game theory is discussed in section 6, entitled *Relationship to von Neumann's theory of games* (Wald, 1945, pp. 279-280), where we read:

> The theory of statistical decision functions which minimize the maximum risk is very closely related to a theory of games developed by John von Neumann [3], [4]. In fact, the problem of statistical inference as formulated can be interpreted as a zero sum two person game in v. Neumann's theory.

The justification for the Minimax paradigm is rather "apologetic" (Wald 1945, p. 279).

> Clearly, the statistician wishes to minimize $r[\theta|\omega(E)]$. Of course, we cannot say that Nature wants to maximize $r[\theta|\omega(E)]$. However, if the statistician is in complete ignorance as to Nature's choice, it is perhaps not unreasonable to base the theory of a proper choice of $\omega(E)$ on the assumption that Nature wants to maximize $r[\theta|\omega(E)]$. Under this assumption a problem of statistical inference becomes identical with a zero sum two person game.

It should be pointed out that the difficulties discussed in Wald (1945, pp. 279-280) regarding the existence of a stable solution for the game do not afflict the generic Maximin/Minimax models used in *decision theory* to analyze problems classified as "decision problem under uncertainty" (e.g. Luce and Raiffa 1957), "decisions under ignorance", (e.g. Resnik 1987), "decisions under strict uncertainty (e.g. French 1988). Here it is assumed that the decision maker plays first and that her decision is known to Nature, so that Nature determines her decision based on the decision selected by the decision maker.

Unfortunately, this important distinction is often ignored to result in a confusion between the "classic" game theory interpretation of the Maximin principle and its "classic" decision theory interpretation.

This was noted already by Luce and Raiffa (1957, p. 279) in their famous book *Games and decisions: introduction and critical survey*:

> The maximin principle can be given another interpretation which, although often misleading in our opinion, is sufficiently prevalent to warrant some comment. According to this view the decision problem is a two-person zero-sum game where the decision maker plays against a diabolical Miss Nature.[1] The maximin strategy is then a best retort against nature's minimax strategy, i.e., against the "least favorable" *a priori* distribution nature can employ. We recall that in a two-person zero-sum game the maximin strategy makes good sense from various points of view: it maximizes 1's security level; and it good against 2's minimax strategy, which there is reason to suspect 2 will employ

> since it optimizes his security level and, in turn, it is good against 1's maximin strategy. In a game against nature, however, such a cyclical reinforcing effect is completely lacking.
>
> Nonetheless, just because a close conceptual parallelism between a d. p. u. u. and a zero-sum game is lacking, it does not follow that the maximin procedure is not a wide criterion to adopt. It has the merit that it is extremely conservative in a context where conservatism *might* make good sense. We will have to say about this later.
>
> [1] In a recent lecture to statisticians one of the authors spoke of "diabolical" Mr. Nature." The audience reaction was so antagonistic that we have elected the path of least resistance.

Note: d. p. u. u. = decision problem under uncertainty; and 1 and 2 refer to Player 1 and Player 2, respectively.

In the context of the generic Maximin model used in this discussion, namely

$$\max_{x\in X}\ \min_{s\in S(x)}\ f(x,s) \tag{5.78}$$

the assumed sequence of moves by the players is implied by the fact that the set of admissible states ($S(x)$) available to Nature (the $\min$ player) depends on $x \in X$.

Chapter 6

Illustrative examples

I conclude the discussion on the main conceptual and technical aspects of decision-making under severe uncertainty with three simple, illustrative examples. They are designed to make vivid the fundamental flaw in the proposition to employ *Radius of Stability* models — such as info-gap's robustness and decision models — to seek decisions that are robust against severe uncertainty of the type stipulated by info-gap decision theory.

6.1 Example 1

The aim of this simple example is three-fold. First, to illustrate that there is a class of robustness problems that are so simple (trivial) that they can be solved by inspection without having to resort to a formal robustness model/analysis. Second, to illustrate that there are cases where there is no difference between local and global robustness. Third, to caution against taking certain properties of such simple problems for generic properties of robustness problems in general.

The main property featured in this example is a *one-dimensional* uncertainty space $\mathscr{U}$, namely the property that the parameter u is a numeric **scalar**. Another important property featured here is that for each $q \in Q$, the performance levels $r(q,u), u \in \mathscr{U}$, are *unimodal* with respect to u.

Consider then the case where $\mathscr{U} = (-\infty, \infty)$, the point estimate is $\tilde{u} = 0$, the critical performance level is $r^* = 2$, the performance requirement is $r^* \leq r(q,u)$, and for a certain decision $q \in Q$ the performance function is defined by

$$r(q,u) := 5 - 0.5|u| \ , \ -\infty \leq u \leq \infty \tag{6.1}$$

Since $r(q,u)$ is strictly decreasing with u, it follows that the set of acceptable values of u is an interval $\mathscr{U}(q) = [\underline{u}, \overline{u}]$ such that $\underline{u} < \tilde{u} \leq \overline{u}$. To obtain the bounds $\underline{u}$ and $\overline{u}$, we solve the equation $r(q,u) = r^*$, obtaining

$$5 - 0.5|u| = 2 \ \longrightarrow \ \underline{u} = -6 \ ; \ \overline{u} = 6 \ ; \ \mathscr{U}(q) = [-6, 6] \tag{6.2}$$

Thus, if we define the *size* of an interval $I = [a, b]$ to be equal to $b - a$, we would have

$$size(\mathscr{U}(q)) = 6 - (-6) = 12 \tag{6.3}$$

This concludes the **global** robustness analysis of decision q. Note that in this analysis we did not stipulate any estimate of the true value of u.

Let us now determine the *Radius of Stability* of decision q.

To do this we need to specify two things: (1) a point estimate, $\tilde{u}$, of the true value of u, and (2) a family of nested balls, $B(\alpha, \tilde{u}), \alpha \geq 0$, around $\tilde{u}$.

But before we do this, it is important to note that the two critical values of u determined by the global robustness analysis of q are also critical within the framework of the local robustness analysis. In particular, assuming that $\underline{u} < \tilde{u} < \overline{u}$, then the *Radius of Stability* of q will be either the length of the interval $[\underline{u}, \tilde{u}]$, or the length of the interval $[\tilde{u}, \overline{u}]$, whichever is smaller, assuming that the recipe for the "distance" to the estimate postulates symmetry. The length of these intervals will depend on the *metric,* or *norm,* used to define the radius of the balls around the estimate $\tilde{u}$.

For example, suppose that $\tilde{u} = 0$. Then, the two intervals under consideration are $[-6, 0]$ and $[0, 6]$ so that if the same scale is used to determine their length, they will have the same length and the *Radius of Stability* will be equal to this length (assuming symmetry). For instance, if we let

$$B(\rho, \tilde{u}) := \{u \in \mathscr{U} : |u - \tilde{u}| \leq \rho\} \ , \ \rho \geq 0 \tag{6.4}$$

we would have

$$\begin{aligned} B(\rho, \tilde{u}) : &= \{u \in \mathscr{U} : |u| \leq \rho\} \ , \ \rho \geq 0 && (6.5) \\ &= \{u \in \mathscr{U} : -\rho \leq u \leq \rho\} && (6.6) \\ &= [-\rho, \rho] && (6.7) \end{aligned}$$

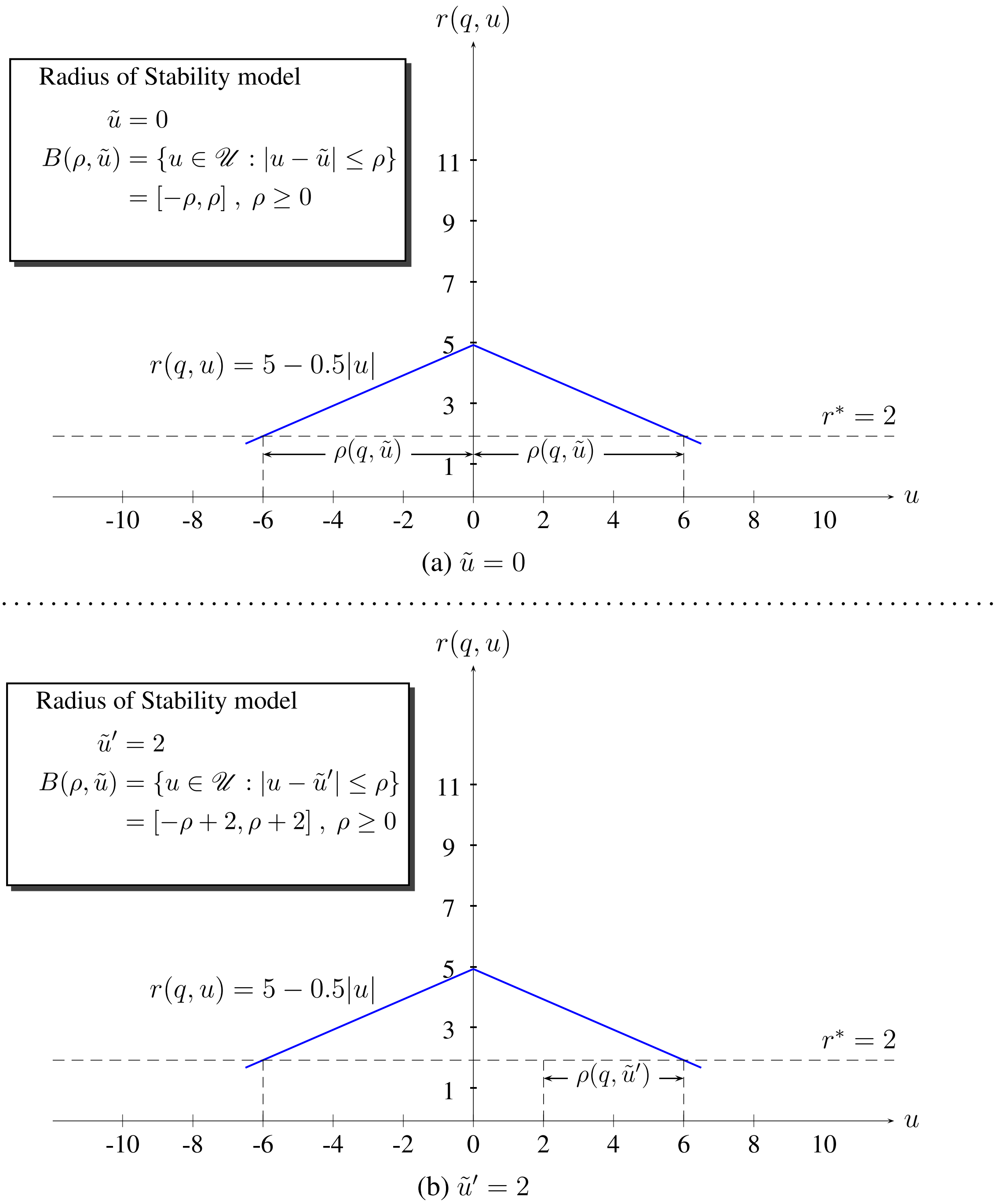
Hence, the largest ball contained in $\mathscr{U}(q) = [-6, 6]$ is that whose radius is equal to $\rho(q, \tilde{u}) = 6$. This is shown in Figure 6.1a.

If we set the estimate to be equal to say $\tilde{u}' = 2$, the two critical intervals will be $[-6, 2]$ and $[2, 6]$ so that the *Radius of Stability* of q will be equal to $\rho(q, \tilde{u}') = 4$. This is shown in Figure 6.1b.

Now, suppose that in constructing the balls we decide to scale the length of the intervals and consider instead balls of the form

$$B(\rho, \tilde{u}') := \left\{u \in \mathscr{U} : \frac{|u - \tilde{u}'|}{\tilde{u}'} \leq \rho\right\} \ , \ \rho \geq 0 \tag{6.8}$$

Then, the same scaling will apply to the *Radius of Stability* of q, namely the scaled *Radius of Stability* of q would be equal to the un-scaled value divided by $\tilde{u}' = 2$, yielding $\rho(q, \tilde{u}')/\tilde{u}' = 4/2 = 2$.



(a) $\tilde{u} = 0$

(b) $\tilde{u}' = 2$

Figure 6.1: Local robustness analysis at $\tilde{u}$ and $\tilde{u}'$

This example illustrates the point that there are *trivial* cases where the (local) *Radius of Stability* analysis generates the set of acceptable values of u. In such cases local robustness coincides with global robustness.

To sum it all up:

- There are cases where a formal model of robustness need not even be contemplated due to the simplicity of the robustness problem in question.
- There are cases where local and global robustness yield the same results.
- Such examples are usually associated with problems where the parameter u is a numeric *scalar* and where the performance function r possesses some *monotonicity* properties.

It is important to appreciate, therefore, that simple problems such as these do not represent "typical" practical robustness problems.

The next example illustrates that even in extremely simple cases, there can still be a vast difference between local and global robustness.

6.2 Example 2

Consider the case where $\mathscr{U} = (-1000, 1000), r^* = 2$, the performance requirement is $r^* \leq r(q, u)$, and Q consists of two decisions q' and q'' whose performance functions are as follows:

$$r(q', u) = \begin{cases} 8 - u & , \quad -\infty < u < 7 \\ 8 & , \quad 7 \leq u \leq 1000 \end{cases} \tag{6.9}$$

$$r(q'', u) = 3 - 0.15|u| \; , \; -\infty < u < \infty \tag{6.10}$$

The corresponding sets of acceptable values of u are then as follows:

$$\mathscr{U}(q') = [-1000, 1000] \setminus (6, 7) \tag{6.11}$$

$$\mathscr{U}(q'') = \left[-6\frac{2}{3}, 6\frac{2}{3}\right] \tag{6.12}$$

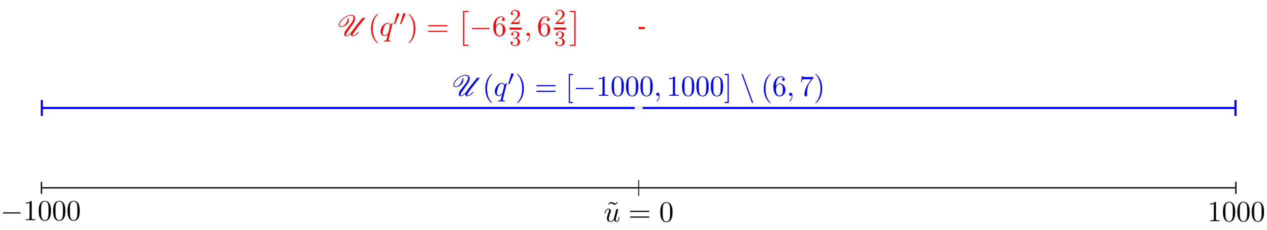
Clearly, according to the *Size Criterion,* decision q' is much more (globally) robust than decision q' on $\mathscr{U} = [-1000, 1000]$.

Thus, the global robustness picture is as shown in Figure 6.2. It displays the uncertainty space $\mathscr{U} = [-1000, 1000]$ and the "acceptable" regions of u associated with the two decisions.

Let us now examine the local robustness of these two decisions, say according to the *Radius of Stability* model.

Note, however, that because the problem is so simple, a formal *Radius of Stability* model is not even required. To illustrate, suppose that the point estimate is $\tilde{u} = 0$. Then for q' the critical interval around the estimate is $I(q') = [0, 6]$. So, if we do not scale u, and use the simple absolute value norm, the *Radius of Stability* of q' would be $\rho(q', \tilde{u}) = 6$.

For q'' the two critical intervals are, $I_+(q'') = [0, 20/3]$ and $I_-(q'') = [-20/3, 0]$. Hence, the *Radius of Stability* of q'' would be $\rho(q'', \tilde{u}) = 20/3$.



Comments:

- The set $\mathscr{U}(q'')$ is too small to be represented properly. It is shown as a red dot.
- The small gap in $\mathscr{U}(q')$ consisting of the interval $(6, 7)$, is too small to be represented properly. It is shown as the much larger white gap in the (blue) line. It you examine careully the (blue) line you should be able to see it.

Figure 6.2: Global robustness of q' and q''

Thus, from the perspective of the *Radius of Stability* model, decision q'' is more robust (locally) at $\tilde{u}$ than decision q'. This is shown in Figure 6.3.

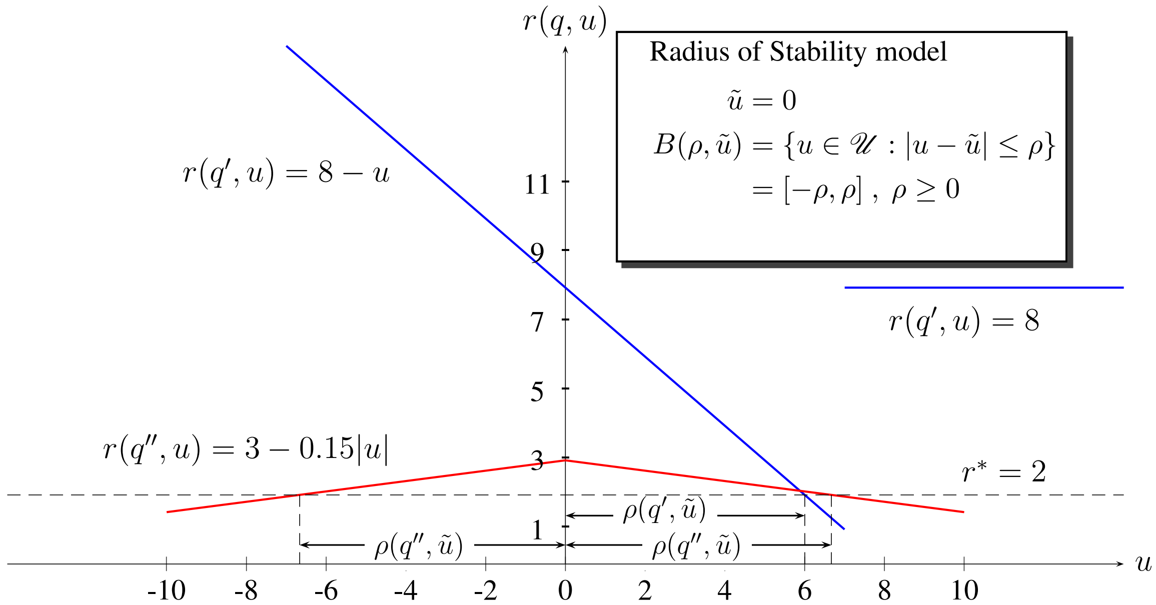


Figure 6.3: Simple illustrative example

Again, I call attention to the fact that there is no contradiction between the results generated by the global and local robustness analyses of the problem:

- Decision q' is much more robust globally (according to the *Size Criterion*) on $\mathscr{U}$ than decision q''.
- Decision q' is (according to the precepts of the *Radius of Stability* model) less robust locally at $\tilde{u}$ than q''.

The *Size Criterion* is a criterion for global robustness, whereas the *Radius of Stability* model is a model of local robustness.

6.3 Example 3

The aim of this example is to illustrate the *No Man's Land* effect of the *Radius of Stability* model so as to make the point that this model is utterly unsuitable for the treatment of severe uncertainty expressed in terms of vast uncertainty spaces.

Consider a case involving two decisions, q' and q'', and a performance constraint of the form $r^* \leq r(q,u)$, where u is a real number whose true value is subject to severe uncertainty and whose point estimate is equal to $\tilde{u} = 0$.

We shall examine the robustness of these decisions from two different perspectives, by addressing the following questions:

- How robust are these decisions in the neighborhood of the estimate $\tilde{u}$?
- How robust are these decisions against the severe uncertainty in the true value of u?

To answer the first question, we use the *Radius of Stability* model. The results are shown in Figure 6.4.

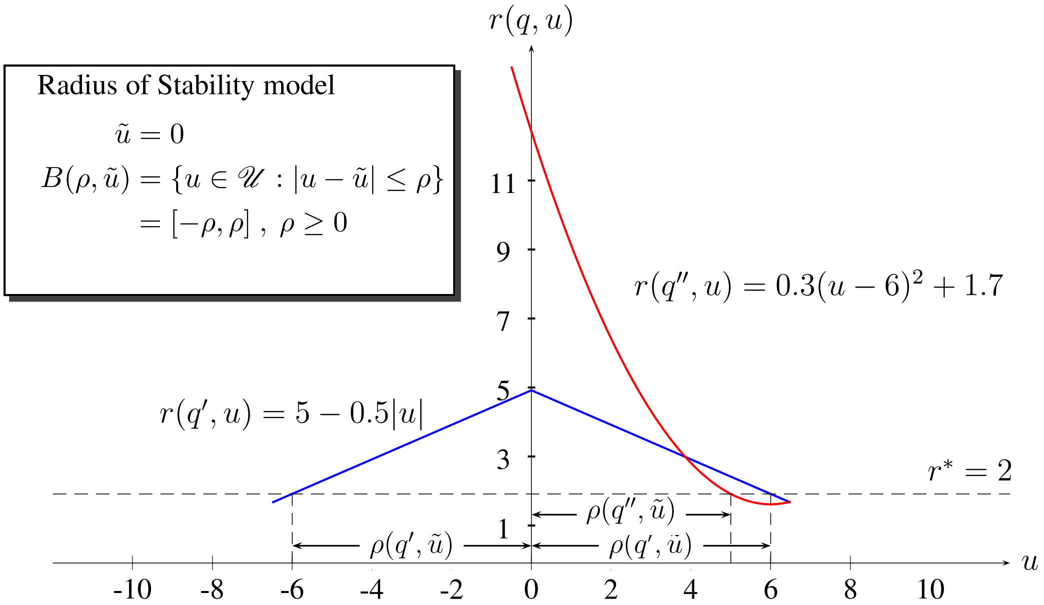


Figure 6.4: Simple *Radius of Stability* model

Thus, according to the *Radius of Stability* model, decision q' is more robust than decision q'', observing that the *Radius of Stability* of q' is equal to 6 whereas the *Radius of Stability* of q'' is equal to 5.

While objections might be raised to this conclusion on the grounds that $r(q'',u) > r(q',u)$ over most of the interval $[-6,6]$, it is not too difficult to see the logic behind this verdict given by the *Radius of Stability* model. In other words, in this case the result generated by the *Radius of Stability* model is most assuredly sensible/reasonable.

Now, back to the issues bearing more directly on severe uncertainty manifested in unbounded uncertainty spaces.

To dramatize the situation let us examine this point in light of Ben-Haim's (2001, 2006) observation that most of the applications of info-gap decision theory involve unbounded uncertainty spaces. Consider then the case where $\mathscr{U} = (-\infty, \infty)$ and the estimate $\tilde{u} = 0$ is a wild guess.

Figure 6.5 shows the result generated by the *Radius of Stability* in this case — which according to the *Invariance Property* are of course identical to the results shown in Figure 6.4.

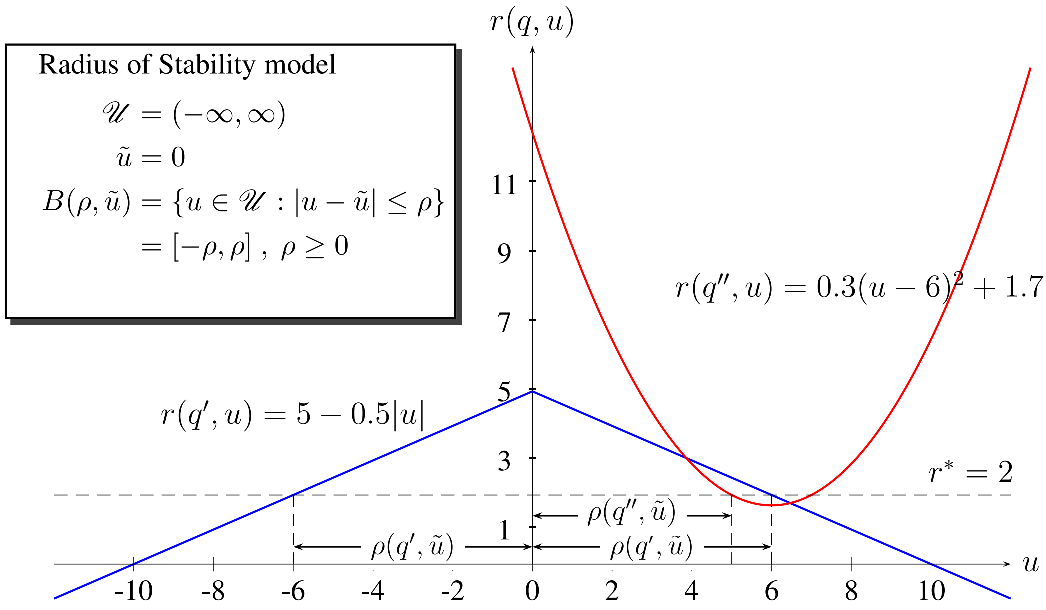


Figure 6.5: Simple *Radius of Stability* model

Observe that the performance level of q'' is better (larger) than the performance level of q' over the entire unbounded uncertainty space $\mathscr{U} = (-\infty, \infty)$ except for a minute interval in the neighborhood of $u = 6$. Furthermore, observe that the difference increases rapidly (quadratically) as u deviates from $u = 6$.

More importantly, note that whereas q'' **satisfies** the performance requirement $r^* \leq r(q'', u)$ over the entire unbounded uncertainty space, except for a minute interval in the neighborhood of $u = 6$, decision q' **violates** the performance requirement $r^* \leq r(q', u)$ over the entire unbounded uncertainty space, except for the minute interval $[-6, 6]$.

Yet, the *Radius of Stability* of q' is equal to $\rho(q', \tilde{u}) = 6$, which is greater than the *Radius of Stability* of q'' which is equal to $\rho(q'', \tilde{u}) = 5$. Thus, as far as the *Radius of Stability* is concerned, q' is more robust than q'' in the neighborhood of the point estimate $\tilde{u} = 0$.

However, as clearly illustrated in Figure 6.5, this does not mean that q' is more robust than q'' over $\mathscr{U} = (-\infty, \infty)$.

Indeed, if we use the *Size Criterion* to determine the decisions' global robustness, we obtain the following sets of acceptable values of u:

$$\mathscr{U}(q'') = (-\infty, \infty) \setminus (5, 7) = (-\infty, 5) \cup [7, \infty) \tag{6.13}$$

$$\mathscr{U}(q') = [-6, 6] \tag{6.14}$$

Thus, from the perspective of the *Size Criterion*, decision q'' is much more robust globally than q' on $\mathscr{U} = (-\infty, \infty)$.

Observe that the local *Radius of Stability* model will generate the same results regardless of how we specify the performance levels $r(q, u)$ for q' and q'' and values of u in the *No Man's Land* specified by, say

$$NML := (-\infty, \infty) \setminus [-6.1, 6.1] = (-\infty, -6.1) \cup (6.1, \infty) \tag{6.15}$$

The *No Man's Land* syndrome associated with this example is shown in Figure 6.6.

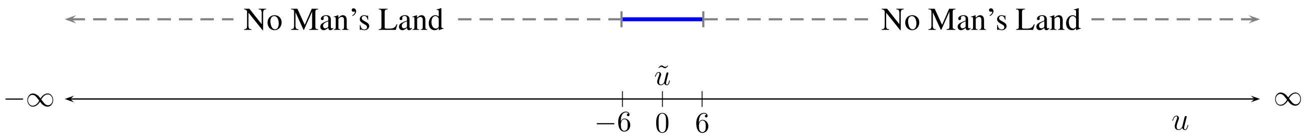


Figure 6.6: Illustration of the No Man's Land Syndrome

It is important to take note that these findings do not find fault with the *Radius of Stability* model *per se*. Rather, they bring out that the *Radius of Stability* model must not be used as a means for determining the global robustness of systems over large, let alone, unbounded, uncertainty spaces because this is not this model's function. As pointed out earlier, the *Radius of Stability* model was devised expressly to provide a means for determining the size of the smallest perturbation in a nominal value of the parameter of interest that can destabilize a system. In this intended capacity the *Radius of Stability* model performs perfectly well.

What this example does demonstrate, though, is that the application of *Radius of Stability* models for the management of severe uncertainty, as advocated by info-gap decision theory, attests to a lack of appreciation of the difference between local and global robustness.

Chapter 7

So what is this all about?

7.1 Introduction

Having detailed the technical aspects of info-gap decision theory against the backdrop of the main methodological and modeling issues in robust decision-making under severe uncertainty, I want to remind the reader of the three main questions that set off this discussion. These were:

· Is info-gap decision theory the theory that it is claimed to be and does it indeed do what it is claimed to do?

· What is the role and place of info-gap decision theory in decision theory and robust decision-making in the face of severe uncertainty?

· What are the implications of the answers to these questions?

As my discussion — notably my technical analysis — clearly demonstrated, a huge gap exists in the info-gap literature between the following two aspects of the theory:

· The **rhetoric** proclaiming what info-gap decision theory is and does.

· The **hard facts** attesting to what it actually is and does.

In greater detail, on the one hand there are statements galore in the info-gap literature asserting that:

· Info-gap decision theory provides a distinct new method for robust decision-making under severe uncertainty, designed specifically for situations where the estimate is poor, the uncertainty space can be vast and the quantification of the uncertainty is likelihood-free.

· Indeed, not only is this methodology claimed to enable a reliable management of severe uncertainty, it is claimed to enable dealing with rare events, catastrophes and surprises.

On the other hand, as we saw:

· Given that info-gap's robustness model is a *Radius of Stability* model, hence a simple instance of Wald's Maximin model, it clearly does not provide a new robustness model. All that is "new" in the methodology put forward by info-gap decision theory, is the misguided proposition that robustness against severe uncertainty can be reliably sought by means of an inherently local analysis of the type conducted by *Radius of Stability* models.

- Because, all that info-gap's robustness model is capable of doing is to seek decisions that are robust against small perturbation in a given nominal value (poor estimate) of the parameter of interest.
- This means that it ignores the performance levels associated with values of the parameter that are outside the ball centered at the estimate whose radius is equal to the *Radius of Stability* of the decision. The inference therefore is that info-gap decision theory takes no account whatsoever of the complexities and difficulties associated with decision-making in the face of severe uncertainty.

The immediate implications of these findings are these:

- Methodologically, info-gap decision theory is utterly unsuitable for decision-making under severe uncertainty of the type that it stipulates. It is unclear, therefore, what role it can have in decision theory and robust optimization.
- Given that its robustness model is a simple *Radius of Stability* model (circa 1960), info-gap decision theory has nothing new to offer in areas where local robustness is the issue.

So the inevitable question is this:

> What is the explanation for the enthusiastic reception accorded to info-gap decision theory by risk analysts in Australia?

I need hardly point out that I do not have a definitive answer to this intriguing question. But, given my experience of the past eight years, I believe that I would be fully justified in claiming that the info-gap *rhetoric,* namely its phraseology, buzzwords and the web of verbiage spun around them, have been central to attracting adherents and sustaining their continued commitment to this theory.

So, to give the reader some idea of the centrality of **rhetoric** in the info-gap discourse, it is important to contrast it with the **basic facts** that were identified in the preceding chapters of this document.

7.2 What then are the basic facts?

Here is, in concentrated form, a list of the basic facts about info-gap decision theory that I discussed in various contexts in the preceding chapters of this document.

The first three facts describe the essential features of info-gap decision theory as a theory whose declared aim is the pursuit of robustness against severe uncertainty.

- **Fact 1:**
  Info-gap decision theory (Ben-Haim 2001, 2006, 2010) is claimed to be a non-probabilistic theory for robust decision-making in the face of severe uncertainty. Its formal robustness model is as follows:

  Info-gap's robustness model:

  $$\hat{\alpha}(q,\tilde{u}) := \max\left\{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\right\}\ ,\ q \in Q \tag{7.1}$$

In words, the robustness of decision q is equal to the largest value of α such that every $u \in U(\alpha, \tilde{u})$ satisfies the performance requirement $r^* \leq r(q,u)$. This definition is illustrated in Figure 7.1, where $U(\alpha,\tilde{u})$ is represented as a circle of radius α centered at $\tilde{u}$ and the shaded area represents the "safe" region of uncertainty $\mathscr{U}(q) := \{u \in \mathscr{U} : r^* \leq r(q,u)\}$, namely the region where $u \in \mathscr{U}$ satisfies the performance constraint $r^* \leq r(q,u)$.

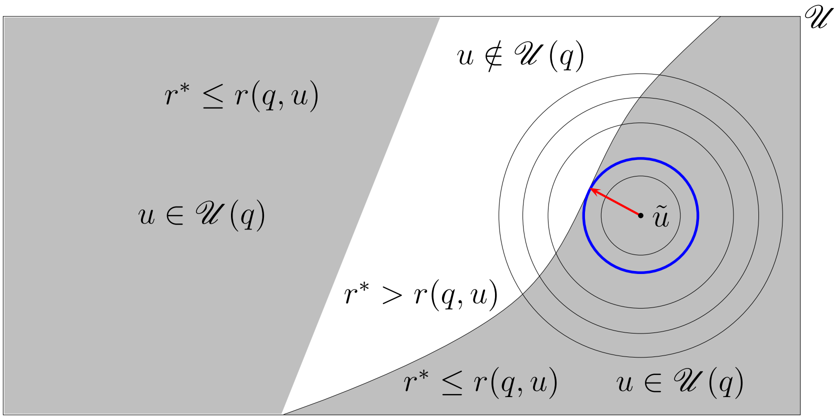


Figure 7.1: Info-gap's robustness of decision q at $\tilde{u}$

Thus, here the robustness of decision q is the radius of the largest circle around $\tilde{u}$ that is contained in the "safe" region of the uncertainty space.

· **Fact 2:**
Info-gap decision theory ranks decisions according to their robustness. Hence, the best (optimal) decision is that whose robustness is the largest. Formally then, info-gap's decision model is the following:

Info-gap's decision model:

$$\hat{\alpha}(\tilde{u}) := \max_{q \in Q} \max \{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\} \tag{7.2}$$

· **Fact 3:**
In the context of info-gap decision theory (Ben-Haim 2001, 2006, 2007, 2010), the severity of the uncertainty is characterized by the following properties:

- The estimate $\tilde{u}$ is a poor indication of the true value of u and it can be substantially wrong. It can be a guess, even a wild guess.
- The uncertainty space $\mathscr{U}$ can be vast, it is often unbounded.
- No likelihood structure is attributed to the uncertainty space.

The next three facts relate info-gap's robustness model to the state of the art in areas ranging from decision theory, control theory, economics, to robust optimization.

· **Fact 4:**
Info-gap's robustness model is a simple *Radius of Stability* model (circa 1960), namely a

model of the form:

Radius of stability model:

$$\hat{\rho}(q, s^*) := \max \{\rho \geq 0 : s \in S_{stable}(q), \forall s \in B(\rho, s^*)\} \ , \ q \in Q \tag{7.3}$$

See Theorem 2.2.1.

The correspondence is established by viewing info-gap's performance requirement $r^* \leq r(q, u)$ as the stability requirement that defines the region of stability $S_{stable}(q)$.

- **Fact 5:**
  The *Radius of Stability* model, hence info-gap's robustness model, are instances of Wald's famous Maximin model (circa 1940), namely instances of generic models such as

  Generic Maximin models:

  $$\max_{x \in X} \min_{s \in S(x)} f(x, s) \tag{7.4}$$

  $$\max_{x \in X} \min_{s \in S(x)} \{f(x, s) : constraints(x, s), \forall s \in S(x)\} \tag{7.5}$$

  $$\max_{x \in X} \{f(x) : constraints(x, s), \forall s \in S(x)\} \tag{7.6}$$

  See Theorem 5.6.1 and Theorem 5.6.2.

- **Fact 6:**
  Info-gap's decision model is not a Maximin model of the reward $r(q, u)$. It is a Maximin model of the horizon of uncertainty α, subject to the performance requirement $r^* \leq r(q, u), \forall u \in U(\alpha, \tilde{u})$.

  See Theorem 5.7.1.

The next four facts are direct implications of the local robustness sought by *Radius of Stability* models.

This is illustrated in Figure 7.2. Observe that according to info-gap decision theory, decision B is more robust than decision C. But, decision C is clearly more robust than decision B against the variability of u over $\mathscr{U}$ — as measured by the "size" of the set of acceptable values of u in $\mathscr{U}$.

- **Fact 7:**
  The *Radius of Stability* model, hence info-gap's robustness model, are models of *local* robustness. That is, they measure the robustness of decisions in the neighborhood of a given value (estimate, nominal value) of the parameter of interest. They are thus invariant with the performance of decisions in areas of the uncertainty space that are outside the largest safe regions of uncertainty of the respective decisions.

- **Fact 8:**
  In view of the inherently local nature of info-gap's robustness model, a decision that is deemed robust (fragile) by info-gap's robustness model is not necessarily robust (fragile) against the severe uncertainty in the true value of u.

- **Fact 9:**

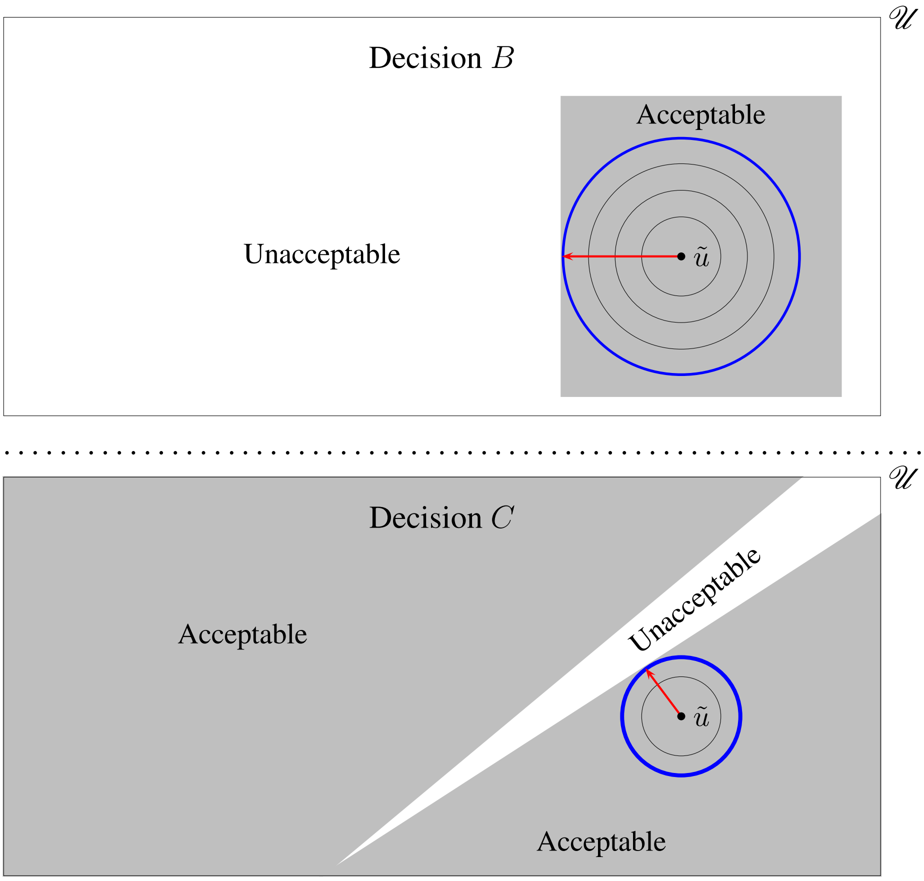


Figure 7.2: Info-gap's robustness analyses of two decisions

Radius of Stability models, hence info-gap's robustness model, do not seek decisions that are robust against severe uncertainty. They seek decisions that are robust against small perturbations in a given value (estimate, nominal value) of the parameter of interest.

- **Fact 10:**
  Info-gap decision theory does not seek a decision with the widest set of acceptable outcomes. It seeks a decision that maximizes the size of the neighborhood around the estimate over which the outcomes are acceptable.

With the list of **hard facts** about info-gap decision theory in front of us, let us now examine more closely some aspects of the **rhetoric** that is used in the info-gap literature to describe its capabilities and its role and place in decision theory.

7.3 Rhetoric

Considering the brouhaha created by Nassim Taleb's two best selling books (*Fooled by Randomness* and *The Black Swan: The Impact of the Highly Improbable*) on severe uncertainty, one would have expected risk analysts to show a greater awareness of the fundamental difficulties presented by severe uncertainty. Recall that one of Taleb's main points is that analysts' reliance on idealized models of reality, dangerously blinds them to the true nature and impact of severe uncertainty, thereby cultivating a false sense of security in these models, which in turn leads to perilously risky policies.

The great caution towards uncertainty, urged by this position, has apparently not penetrated the info-gap literature.

To the contrary, reading this literature one is struck by the great confidence shown in info-gap's capabilities to take on rare events, surprises, shocks, catastrophes (e.g. Ben-Haim 2001, 2006, 2010), and even ... "unknowns unknowns" and Black Swans! (e.g. Wintle et al. 2010).

The reason for this misplaced confidence in the capabilities of info-gap decision theory is apparently due to the *face value* acceptance, more precisely the uncritical acceptance, of the proposition that the primary objective of this theory is the pursuit of **robustness to severe uncertainty**.

The reasoning underlying this assessment of info-gap decision theory is based on the following seemingly sound logical progression:

· Info-gap's robustness analysis ranks decisions on the basis of their robustness against severe uncertainty: the larger the robustness, the better.

· Accordingly, the best (optimal) decision is that whose robustness to severe uncertainty is the largest.

· Surely, this cannot be bettered!

· The implication therefore is that info-gap decision theory is precisely the theory for obtaining robustness to severe uncertainty.

I hasten to add that this logical progression is not spelled out in so many words. Nevertheless, it is clearly discernible between the lines of statements such as this (emphasis added):

> An extension of the current study would be to determine the optimal management effort under uncertainty of the density-impact curve by, for example, assuming a probability distribution for the parameters of the density-impact relationship or information-gap decision theory (Ben-Haim 2001). **Information-gap decision theory derives the most robust management option to meet a minimum performance requirement under severe uncertainty** (Ben-Haim 2001, Regan et al. 2005).
>
> Yokomizo et al. (2009, p. 384)

As I indicated in the preceding chapters, such unfounded descriptions/assessments of info-gap decision theory unjustifiably portray it as a theory of *global* rather than *local* robustness. And the trouble is that this misrepresentation is further reinforced by attributions of capabilities to info-gap's robustness analysis that it cannot (by definition) have. For instance, the capability to *maximize the likelihood* or *chance* or *reliability* of acceptable performance, or the capability to generate a decision that yields the *widest range* of acceptable outcomes.

My point is — and this is one of the important lessons of the info-gap experience — that the rhetoric of an informal discussion of the properties, capabilities, mode of operation, and so on, of a mathematical model must remain faithful to ... the definition of the model.

In the case of info-gap decision theory, the robustness model under consideration is a simple instance of the well known *Radius of Stability model,* hence of Wald's famous Maximin model. These are simple, well-established, well understood-models, which means of course that the same is true about info-gap's decision and robustness models. They are simple, well-understood models.

This means that for an informal discussion about the capabilities of info-gap decision theory, its mode of operation, etc. to have any merit, indeed any validity, one must make it clear that the discussion is about these models. This means that any meaningful discussion on what info-gap decision theory **is** or **is not** must be conducted within the framework of these models, and not on the basis of some general, ambiguous declarations that are bandied about in the abstract.

For instance, assertions such as "info-gap is not a worst case analysis", or "info-gap explores the full space of monitoring investment options and parameter uncertainties" must be proved in the context of these models. In particular, it is important to make sure that interpretations of these models do not contradict the models themselves, the axioms on which they are based, and so on.

And it should be helpful to convey the meaning and functions of these models graphically through illustrations such as Figure 7.1 and Figure 7.2 to enable a more "intuitive" grasp of the models and their properties.

The bottom line is then that no amount of rhetoric that is disconnected from these mathematical models can explains what *info-gap decision theory* is and does. No amount of rhetoric can fix the flaws in the theory. And no amount of rhetoric will be able to meet the criticism directed at the theory. Critical claims about the theory must be dealt with in the context of these mathematical models.

The following examples illustrate the role of rhetoric in ascribing info-gap decision theory capabilities that it does not have. Note that the first three have a significant local (Australian) content.

7.3.1 Example

In a recent article, Wintle et al. (2011) make the astounding claim that info-gap decision theory **generalizes** Wald's Maximin strategy. This claim is astounding because: how can info-gap decision theory possibly generalize Wald's Maximin strategy when info-gap's robustness model is an *instance* of Wald's Maximin model? (see Appendix J for further details).

7.3.2 Example

Consider the following statement, quoted from a paper entitled *Allocating monitoring effort in the face of unknown unknowns,* that featured as the cover story in a recent issue of Decision Point (issue 43, 2010):

> The third type of model application would involve a formal uncertainty analysis that explores the full space of monitoring investment options and parameter uncertainties to identify the most robust monitoring investment (*sensu* Wald 1945, Ben-Haim 2006). A formal uncertainty analysis would identify the robust-optimal monitoring investment

Source: The picture is a NASA satellite image of Australia. See WIKIPIDIA at: http://en.wikipedia.org/wiki/File:Australia_satellite_plane.jpg.

Figure 7.3: A global view of an island

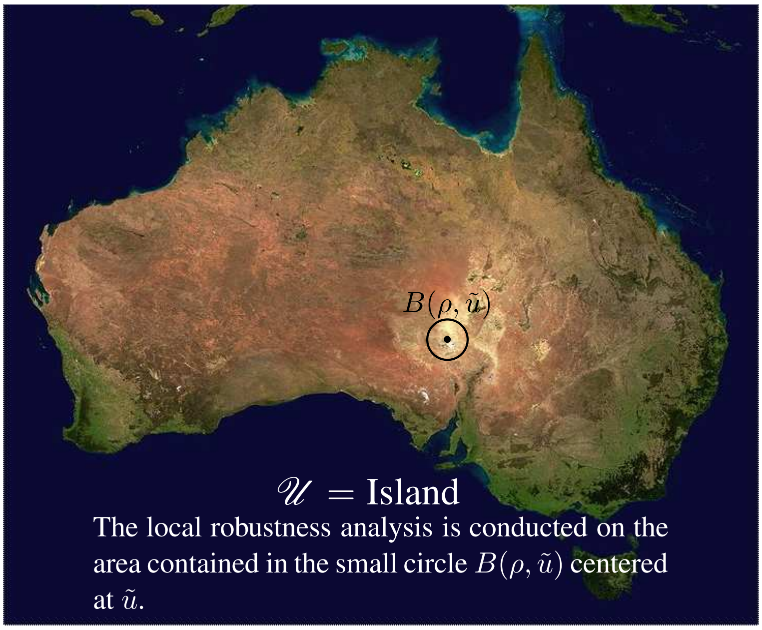
> strategy that achieves some minimum performance criteria under the most extreme scenario of parameter estimation error, or the widest range of possible parameter values, depending on the preferred definition of robustness.
>
> Wintle et al. (2010, p. 8)

In this article, published in *Ecology Letters,* info-gap decision theory figures prominently alongside Wald's Maximin model, as a methodology that is capable of dealing with severe uncertainty of the type associated with *unknown unknowns* and Black Swans. As indicated by the quote, this capability is apparently attributed to info-gap decision theory on grounds of its purported inherent ability to "...explore the full space of monitoring investment options and parameter uncertainties to identify the most robust monitoring investment ...".

The mental picture that you would probably conjure up reading such a statement would be that of a large island, such as the one shown in Figure 7.3, where the parameter of interest would be a location on the island whose "true whereabouts" is subject to severe uncertainty. Namely, all that is known about the location (true value of the parameter) is that it can be just about anywhere in the island.

Now, risk analysts who are well-versed in robustness analysis but are not familiar with info-gap decision theory, would immediately conclude that info-gap's robustness model is a model of **global** robustness. That is, a model that determines robustness through an exploration of the entire parameter space to yield decisions whose set of acceptable outcomes is the largest (or something along these lines).

However, once shown the simple illustration displayed in Figure 7.4, even analysts who are unaware of the huge gap between the rhetoric and the basic facts about info-gap's robustness model, can immediately see how erroneous this rhetoric is. Indeed, you need not even be a risk analyst to see why this rhetoric is grossly misleading.



The island represents the uncertainty space $\mathscr{U}$, and the small circle the "sample space" on which the robustness analysis is conducted. This is the area surrounding Elliot Price Conservation Park.

Figure 7.4: Fine print of info-gap's (local) robustness analysis

For the record, I should point out that illustrations such as this have been on my website since the end of 2006. Such illustrations have also featured prominently in my many presentations and working papers on info-gap decision theory. And yet, four years later, we are still reading unsubstantiated rhetoric about info-gap's robustness analysis exploring the entire uncertainty space.

What is the wonder then that my criticism of info-gap decision theory, which exposes the errors in such statements, is considered "too harsh" by some info-gap scholars!

Remark

My point of course is that the trouble in the info-gap literature is not that the description of properties and features of mathematical models is done verbally. Indeed, there is no doubt that concepts such as *local robustness* can be put across clearly and unambiguously in plain English, or for that matter any other language. And to illustrate, consider the following opening sentences in the abstract of a paper entitled *Local robustness analysis: Theory and application* by Brock and Durlauf (2005, p. 2067):

> This paper develops a general framework for conducting local robustness analysis. By local robustness, we refer to the calculation of control solutions that are optimal against the least favorable model among models close to an initial baseline.

Although the details of the robustness model are not specified in this short paragraph, the fact that the model in question is a model of *local* robustness is stated, and the model is profiled for what it is: a model of local robustness.

But, as explained in this document, in the case of info-gap decision theory, the problem is far more fundamental than the terminology. It is not simply that the language is inaccurate. The language in fact betrays a complete lack of awareness of the distinction between *local* and *global* robustness. Consequently, info-gap publications give a totally misleading impression about the results that they provide.

The following is another example of the total incongruity between the rhetoric about info-gap's robustness analysis and the basic facts.

7.3.3 Example

Great stress is laid in info-gap's primary texts, and if you will, its mainstream literature, on the fact that info-gap decision theory is a **non-probabilistic, likelihood-free** theory. Indeed, info-gap's robustness model, hence info-gap's decision model, are claimed to be devoid of notions such as likelihood, chance, beliefs, and so on.

This means that the measure of "distance" provided by the nested neighborhood structure $U(\alpha, \tilde{u}), \alpha \geq 0$, does not in any way represent any notions of probability, likelihood, chance, beliefs and so on. And this means of course that this measure of "distance" cannot/must not be used for this purpose.

Info-gap's main texts (e.g. Ben-Haim 2001, 2006) are crystal clear about this point. After all, the great innovation of info-gap decision theory, is supposed to be in its being ... non-probabilistic and likelihood-free.

But this does not stop info-gap scholars from resorting to a rhetoric, which in fact ascribes info-gap decision theory capabilities that it does not have, such as those described in the following quotes (emphasis added):

> Rather than specifying the extent of uncertainty in parameters at the outset, info-gap theory takes the position that the best strategy is the one that gives us an outcome that is both acceptable and keeps us immune from unacceptable outcomes given some level of uncertainty (Ben-Haim 2001). **That is, we choose a strategy that maximizes the reliability of an adequate outcome** (i.e. an acceptable value for the persistence criterion, q).
>
> Halpern et al. (2006, pp. 5-6)

> The decision may not minimize the extinction risk when uncertainty is ignored, but it is **the option least likely to fail because of uncertainty in model structure or parameter estimates.**
>
> Nicholson and Possingham (2007, p. 252)

> Information-gap (henceforth termed 'info-gap') theory was invented to assist decision-making when there are substantial knowledge gaps and when probabilistic models of uncertainty are unreliable (Ben-Haim 2006). In general terms, **info-gap theory seeks**

> **decisions that are most likely to achieve a minimally acceptable (satisfactory) outcome in the face of uncertainty,** termed robust satisficing. It provides a platform for comprehensive sensitivity analysis relevant to a decision.
>
> Burgman et al. (2008, p. 8)

> As the horizon of uncertainty α gets larger, the sets $U(\alpha, \tilde{u})$ become more inclusive. **The info-gap model expresses the decision maker's beliefs about uncertain variation of u around $\tilde{u}$.**
>
> Davidovitch et al. (2009, p. 4)

> Info-gap theory identifies as the best policy the one that is most robustly satisfying (Ben-Haim, 2006), i.e. the goal is not to minimize the NPV of total costs but to **maximize the reliability of an acceptable outcome.**
>
> Carrasco et al. (2010, p. 532)

> The Precautionary Principle is supposed to apply under conditions of severe uncertainty, and to maintain a high level of environmental protection. Given these objectives, the principle should be understood as imposing a robust satisficing approach to environmental management. That is, **our decision model should aim at maximizing the chance of an acceptable outcome, and our conclusions should be robust against potential errors in the underlying scientific model.** So, rather than estimating precise payoffs, this approach would be more about classifying outcomes into acceptable vs. unacceptable or manageable vs. unmanageable. For many environmental policy problems — species conservation, global warming, intactness of ecosystems — such an approach is probably more appropriate, and also potentially more feasible.
>
> Sprenger (2011, p. 7)

Compare these to repeated warnings such as these (emphasis added):

> However, unlike in a probabilistic analysis, r has **no connotation of likelihood.** We have no rigorous basis for evaluating how likely failure may be; we simply lack the information, and to make a judgment would be **deceptive** and could be **dangerous.** There may definitely be a likelihood of failure associated with any given radial tolerance. However, the available information does **not** allow one to assess this likelihood with any reasonable accuracy.
>
> Ben-Haim (1994, p. 152)

> Uncertainty is the potential for deviation of an actual realization from its normative form. Neither norm nor any specific potential realization is uncertain; it is the potential for deviation of one from the other which is info-gap uncertainty.
>
> The spatial analogy for info-gap uncertainty demonstrates that we need **no concept of chance, frequency of recurrence, likelihood, plausibility or belief in order to speak of uncertainty.**
>
> Ben Haim (2006, p. 22)

The trouble is, however, as indicated above, that such warnings are unlikely to bear fruit because of the irreconcilable contradiction between two basic facts. That is, the fact that info-gap's robustness model is likelihood-free runs counter to the central role that the estimate $\tilde{u}$ play in the info-gap robustness analysis. Given this setting, it is apparently very tempting, perhaps even irresistible, to fill in the "vacuum" with a (presumably private) "intuitive", likelihood structure, so as to justify the local approach to robustness prescribed by info-gap decision theory. I discuss this issue in §3.11.

And another example of the total incongruity between the rhetoric about info-gap's robustness analysis and the basic facts.

7.3.4 Example

There is a widespread (erroneous) view in the info-gap literature that info-gap's robustness analysis seeks a decision whose set of acceptable outcomes is the largest. In fact, the recent proposition by Schwartz et al. (2011) to use "robust satisficing" (read: info-gap decision theory) as a normative standard of rational decision making, is based, in large part, on this (erroneous) assessment of info-gap robustness (emphasis added):

> That is, robust satisficing asks, "what is a 'good enough' outcome," and then seeks the option that will produce such an outcome under the **widest set of circumstances.**
>
> Schwartz et al. (2011, p. 1)

> The robust satisficer answers two questions: first, what will be a "good enough" or satisfactory outcome; and second, of the options that will produce a good enough outcome, which one will do so under the **widest range of possible future states of the world.**
>
> ibid, pp. 9-10

> A robust satisficing decision (perhaps about pollution abatement) is one whose outcome is acceptable for the **widest range of possible errors in the best estimate.** No probability is presumed or employed.
>
> ibid, p. 19

> For an individual who recognizes the costliness of decision making, and who identifies adequate (as opposed to extreme) gains that must be attained, a satisficing approach will achieve those gains for the **widest range of contingencies.**
>
> ibid, p. 27

Similar assertions are made elsewhere, for instance (emphasis added):

> The robust-satisficing strategy chooses an allocation that guarantees an acceptable total crime rate (which usually will not be the estimated minimum) for the **largest possible range of error in the estimated elasticities.**
>
> Davidovitch and Ben-Haim (2011, p. 13)

But the fact, of course, is that info-gap robustness is **not** a measure of the "size of the set of acceptable values of u". Rather, it gives the size of the largest neighborhood around the estimate $\tilde{u}$ all of whose elements are "acceptable".

It is the *Size Criterion* (section 2.3) that measures the "size" of the set of acceptable values of u. Thus, Schwartz et al.'s (2011) assessment of info-gap decision theory is based on a capability that it **does not possess**. This is illustrated graphically in Figure 7.2 in connection with the discussion in **Fact 7-10** on the local nature of info-gap's robustness model.

And finally, a particularly edifying example of the discrepancy between the rhetoric and the facts, can be found in the claims or propositions that info-gap's robustness model can be a proxy to "probability of success" models.

7.3.5 Example

There is an ongoing discussion in the info-gap literature on the formulation of "proxy theorems", that is theorems stipulating conditions under which info-gap's robustness model acts as a proxy to "probability of success" models. The idea here is that, under these conditions, info-gap's ranking of decisions effectively yields a ranking based on their probability to generate an acceptable outcome (e.g. Ben-Haim 2007, 2007a, 2009, Davidovitch 2009, Ben-Haim and Cogan 2011).

For example, consider this:

> We show that the non-probabilistic info-gap robustness function can be used to choose a computational linear model for which the probability of bounding the non-linear model is maximized, without knowing the probability distribution of the parameters of the non-linear model.
>
> Ben-Haim and Cogan (2011, p. 14)

Of course, to see how far-fetched such a proposition is one need not even bother to add up all the hard facts about info-gap decision theory that are relevant to this issue. Still, considering that this proposition is repeated in a number of recent publications, and considering the significance that seems to be attributed to it, it is important to make it explicit why this proposition is far-fetched. To this end, simply keep in mind that info-gap decision theory imposes no requirement whatsoever on the uncertainty space $\mathscr{U}$, nor on the performance function r. Add to this the fact that, info-gap's uncertainty model is non-probabilistic and likelihood-free, and the fact that its robustness model is by definition local, and it is eminently clear that:

- Methodologically, info-gap robustness can hardly be a "proxy" to probability of success.
- Furthermore, that such a proposition can be true only in rare cases which by necessity (namely, methodologically) would be "trivial".

And to be sure, as I show in Appendix E, the robustness problem studied in Ben-Haim and Cogan (2011) is so trivially simple that its solution literally stares you in the face. This means that the (global) robustness of the problem under consideration is determined by inspection directly from the problem's formulation. And the implication is of course that given the problem's trivial simplicity, formulating a formal robustness model is rendered utterly unnecessary.

In sum, one has to exercise great caution when reading statements made in the info-gap literature about the capabilities of the theory (see discussion in Appendix D.3).

7.4 FAQs about info-gap decision theory

I want to point out that while the above short list of basic facts about info-gap decision theory (preceding my discussion on rhetoric) gives a comprehensive picture of the theory, many questions which require attention are not addressed by them.

Readers can consult the compilation of FAQs about info-gap decision theory on my website. However, to make this manuscript self-contained, I provide a short compilation of FAQs about info-gap decision theory in the appendix. The list in Appendix G addresses the FAQs posed by Ben-Haim (2007), whereas the list in Appendix H includes some of the FAQs posted on my website[1].

7.5 Reviews of info-gap publications

And as a final note, I want to add that in 2009 I began posting on my website (my) reviews of publications on info-gap decision theory. The directory[2] now includes reviews of 33 publications, most of which are articles published in peer reviewed journals. Of particular relevance to the discussion in this document are the reviews of the following articles:

- Beresford-Smith, B., and Thompson, C.J. (2009) An info-gap approach to managing portfolios of assets with uncertain returns. *Journal of Risk Finance,* 10(3), 277-287.
- Burgman, M.A. (2008) Shakespeare, Wald and decision making under uncertainty. *Decision Point,* 23, 8.
- Davidovitch, L., Stoklosa, R., Majer, J., Nietrzeba, A., Whittle, P., Mengersen, K., and Ben-Haim, Y. (2009) Info-Gap theory and robust design of surveillance for invasive species: The case study of Barrow Island.*Journal of Environmental Management,* 90(8), 2785-2793.
- Moilanen, A., Runge, M.C., Elith, J., Tyre, A., Carmel, Y., Fegraus, E., Wintle, B., Burgman, M., and Ben-Haim, Y. (2006b) Planning for robust reserve networks using uncertainty analysis. *Ecological Modelling,* 199(1), 115-124.
- Regan, H.M., Ben-Haim, Y., Langford, B., Wilson, W.G., Lundberg, P., Andelman, S.J., Burgman, M.A., (2005) Robust decision making under severe uncertainty for conservation management. *Ecological Applications,* 15(4), 1471-1477.
- Rout, T.M., Thompson, C.J., and McCarthy, M.A. (2009) Robust decisions for declaring eradication of invasive species. *Journal of Applied Ecology,* 46, 782-786.
- Sprenger J. (2011) The Precautionary Approach and the Role of Scientists in Environmental Decision-Making. Presented at the Philosophy of Science Association (PSA) 2010 Conference, November 4?6, 2010, Montréal, Quebec, Canada. http://www.laeuferpaar.de/Papers/PSA_Symposium_Paper_v3.pdf (January 9, 2011).
- Sprenger J. (2011) Precaution with the Precautionary Principle: How does it help in making decisions. *Decision Point,* 48, 7.

[1] See http://info-gap.moshe-online.com/faqs.html

[2] See http://info-gap.moshe-online.com/reviews.html.

· Wintle, B.A., Runge, M.C., and Bekessy, S.A. (2010) Allocating monitoring effort in the face of unknown unknowns. *Ecology Letters,* 13(11), 1325-1337.

· Wintle, B.A., Bekessy, S.A., Keith, D.A., van Wilgen, B.W., Cabeza, M., Schroder, B., Carvalho, S.B., Falcucci, A., Maiorano, L., Regan, T.J., Rondinini, C., Boitani, L. and Possingham, H.P. (2011) Ecological-economic optimization of biodiversity conservation under climate change. *Nature Climate Change,* Volume 1, 355-359.

· Yokomizo, H., Possingham, H.P., Thomas, M.B., and Buckley, Y. M. (2009) Managing the impact of invasive species: the value of knowing the density-impact curve. *Ecological Applications,* 19(2), 376-386.

Info-gap scholars who contemplate writing about the theory would do well to browse through the reviews to learn more about how to avoid the pitfalls catalogued in this document.

Chapter 8

Conclusions and recommendations

The most obvious conclusion to be drawn from this discussion is that info-gap decision theory is, to put it mildly, highly problematic, both as a methodology and as a practical tool.

- As a methodology, it fails to address the complexities and difficulties encountered in the treatment of severe uncertainty of the type that it stipulates. Its simplistic recipe, which prescribes focusing on the perturbations in the value of a point estimate, effectively **ignores** the severity of the uncertainty.
- As a practical tool, info-gap's robustness model is none other than a *Radius of Stability* model (circa 1960). It follows therefore that nothing is to be gained from turning to info-gap decision theory, because there is nothing that this model can do that cannot be done with the good old *Radius of Stability model.* In fact, there is a lot to be lost, because as I need hardly point out, info-gap's robustness model lacks the knowledge-base and technical infrastructure (e.g. literature, applications, algorithms, software) that back up the *Radius of Stability* model.
- In fact, given its isolation from the state-of-the-art, info-gap decision theory deprives its users access to the vast literature on *Radius of Stability* models and theories that do address the complexities and difficulties associated with severe uncertainty, e.g. theories developed in the field of *Robust Optimization*.

My recommendations are then as follows:

- **Treatment of severe uncertainty:**
  Info-gap decision theory is utterly unsuitable for the treatment of severe uncertainty. It should therefore not be used for this purpose. The main discipline in the area of decision-making that addresses the complexities and difficulties associated with robust decision-making in the face of severe uncertainty is *Robust Optimization.*
- **Treatment of small perturbations in a nominal value:**
  Because info-gap's robustness model is a reinvention of the *Radius of Stability* model, it must be assessed in relation to areas of expertise where this model is used. The traditional terminology and established conceptual models used for decades to describe the intuitive notion "stability/robustness" provide a far more accurate account of this concept, than the terminology and models provided by info-gap decision theory. Thus, there seems to be no

point in using info-gap decision theory as a framework for the description of the concept that is universally known as "radius of stability".

- **Incorporating likelihood structures in *Radius of Stability* models:**
  It is not too difficult to incorporate likelihood structures in *Radius of Stability* models such as info-gap's robustness model. However, this should be done properly and in a manner that is consistent with the structures that are already part of the model. This in turn may require an adjustment in the terminology used.
- **Conceptualizing info-gap's robustness model:**
  Extensive experience over the past 60 years has shown that robustness issues of the type addressed by info-gap decision theory are best conceptualized as a *2-player game*: The decision maker against Nature, where Nature plays the role of *Uncertainty.* Mathematically, this lends itself to a *Maximin game* — representing robustness, and a *Minimin game* representing opportuneness.
- **Conceptualizing info-gap's local robustness analysis:**
  In view of the prevailing mistaken view in the info-gap literature that info-gap's robustness analysis explores the entire uncertainty space, it is instructive to conceptualize info-gap's local robustness analysis and the resulting *No Man's Land* phenomenon by means of the *Inflated Balloon model* described in Appendix B.
- **Users of info-gap decision theory:**
  Scholars/analysts who nevertheless decide to use info-gap decision theory as a framework for a local robustness analysis, should face up to what info-gap decision theory is and does and what it is not and does not do. More than anything else, they should avoid using a phraseology that misrepresents the fact that info-gap's robustness model is a model of local robustness, indeed, a simple *Radius of Stability* model, hence a simple instance of Wald's maximin model.

Decision-making in the face of severe uncertainty is a daunting task that requires a careful analysis. It is naive in the extreme to suggest that this task can be reliably accomplished by examining the local robustness of decisions in the neighborhood of a wild guess of the true value of the parameter of interest.

Universally accepted maxims such as

- Garbage in – Garbage out!

and

- The results of an analysis can only be as good as the estimates on which they are based!

were formulated precisely to warn against simplistic "too good to be true'' theories that presumably tackle such difficult tasks. These maxims should therefore be kept in mind to avoid using models of local robustness for the management of severe uncertainty.

To be constructive, I call attention to the rich literature on this subject which reports on the tremendous progress over the past fifty years in this area of decision theory. Admittedly, since this literature requires technical/specialized expertise, it may not be easily accessible to all scholars/analysts in such areas as applied ecology, conservation biology, environmental management, etc.

This is the challenge facing ACERA's scholars/analysts: a successful *transfer of knowledge* from relevant fields such as control theory, decision theory and robust optimization, to applied ecology, conservation biology, environmental management, bio-security, etc.

The contribution of the discussion in this document is in the light it sheds on these two facts:

· The field of *decision-making in the face of severe uncertainty* is a well-established area of expertise with an enormous knowledge base.

· A lack of familiarity with this field may result in the reinvention of wheels, perhaps even square ones!

And to sum it all up:

The enthusiasm with which info-gap decision theory was received in ACERA seems to reflect the growing interest in ACERA in methods for decision-making in the face of severe uncertainty. But as indicated in this document, the use of info-gap decision theory for this purpose is very problematic.

It is therefore important that scholars/analysts who use this theory take note of its flaws and of its role and place in decision-making in the face of severe uncertainty.

This document provides an easily accessible starting point for scholars and analysts who seek a rigorous assessment of info-gap decision theory as a methodology and as a practical tool for robust decision-making in the face of severe uncertainty.

It should be read and assessed for what it is: a constructive, comprehensive critique of info-gap decision theory compiled specifically for info-gap scholars in the *Land of the Black Swan.*

And, as a final note.

I want to acknowledge with appreciation the CSIRO recent report *Uncertainty and Uncertainty Analysis methods* (Hayes, 2011). I need hardly point out that I am most gratified that scholars at the CSIRO decided to embark on this study, which inter alia, alerts analysts in applied ecology and conservation biology to pitfalls associated with info-gap decision theory. I trust that readers of my document will recognize that my criticism of info-gap decision theory is no longer a "lone voice calling in the wilderness". I therefore urge readers of this document to read carefully the CSIRO REPORT and my comments on this report posted on my website[1].

[1] See http://info-gap.moshe-online.com/csiro.html

Appendices

Appendix A

Balls

In mathematics, a *ball* or *neighborhood*, is a set of objects that are within a certain "distance" from a given nominal object — the "center" of the ball or neighborhood. Thus, we let $B(\rho, c)$ denote a ball of radius ρ centered at c. More formally, we define such a ball as follows:

$$B(\rho, c) := \{b \in \mathcal{B} : dist(b, c) \leq \rho\} \ , \ \rho \geq 0, c \in \mathcal{B} \tag{A.1}$$

where $\mathcal{B}$ denotes the set of all the objects under consideration, and $dist(b, c)$ denotes the distance between the two objects $b, c \in \mathcal{B}$.

The definition of $dist(b, c)$ can vary, depending on the application, but it invariably has the following two properties:

$$B(0, c) = \{c\} \tag{A.2}$$

$$B(\rho, c) \subseteq B(\rho + \varepsilon, c) \ , \ \forall \rho, \varepsilon \geq 0 \tag{A.3}$$

The first is an implication of the property that the distance between any point $b \in \mathcal{B}$ to itself is equal to zero, namely $dist(b, b) = 0, \forall b \in \mathcal{B}$. The second is a *set containment property* — called "nesting" in info-gap decision theory — that requires $B(\rho, c)$ to be "non-decreasing" (set-containment-wise) with the radius ρ:

$$\rho' \leq \rho'' \longrightarrow B(\rho', c) \subseteq B(\rho'', c) \tag{A.4}$$

The distance function, $dist$, is usually a *metric* or a *norm* consistent with $\mathcal{B}$.

In some applications it is more convenient to consider "open", rather than "closed" balls, in which case we have

$$B(\rho, c) := \{b \in \mathcal{B} : dist(b, c) < \rho\} \ , \ \rho \geq 0, c \in \mathcal{B} \tag{A.5}$$

observing that here $B(0, c)$ is the empty set. In this document the balls are assumed to be closed.

Note that according to the above, a ball does not necessarily have to be circular. For example, the ball

$$B(\rho, c) := \left\{ b \in \mathcal{B} : \rho \geq \max_{i=1,2,3} |b_i - c_i| \right\} \ , \ \rho \geq 0, c \in \mathcal{B} := \mathbb{R}^3 \tag{A.6}$$

$$= \{ b \in \mathcal{B} : |b_1 - c_1| \leq \rho, |b_2 - c_2| \leq \rho, |b_3 - c_3| \leq \rho \} \tag{A.7}$$

is a cube of side 2ρ in $\mathbb{R}^3$ centered at $c = (c_1, c_2, c_3) \in \mathbb{R}^3$.

Note that the implied "distance" function in this case is defined as follows:

$$dist(b, c) := \max_{i=1,2,3} |b_i - c_i| \ , \ b, c \in \mathbb{R}^3 \tag{A.8}$$

By the same token, the center point c does not necessarily have to be the real "center" of the ball, as illustrated in Figure A.1, where the balls are rectangles.

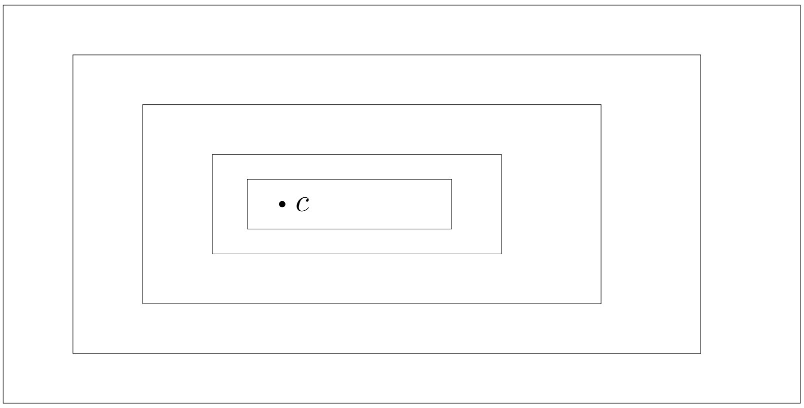


Figure A.1: Nested rectangular balls

This means that theoretically there is a great deal of leeway in the construction of balls. The question is: *what type of ball should be used in a given application?*

This is an important modeling issue. It is particularly important in the modeling of severe uncertainty by means of likelihood-free models. Because, in this framework the decision as to what type of ball would be used is often a highly subjective matter. The point is then that this choice may have a profound impact on the results generated by the analysis.

Surprisingly, info-gap decision theory (Ben-Haim 2001, 2006, 2010) gives not the slightest indication that its "nested" regions of uncertainty are what are known in mathematics, and related areas such as engineering, as "balls" or "neighborhoods".

In the language that in mathematics is applied to the treatment of "balls" and "distances", the term "info-gap" would represent the (unknown) distance of the (unknown) true value of the parameter of interest from its (known) point estimate. In plain language, it is the deviation of the estimate from the true value.

Appendix B

Small perturbations

According to the *American Heritage Science Dictionary:*

perturbation

1. A small change in a physical system, most often in a physical system at equilibrium that is disturbed from the outside.
2. Variation in a designated orbit, as of a planet, that results from the influence of one or more external bodies. Gravitational attraction between planets can cause perturbations and cause a planet to deviate from its expected orbit. Perturbations in Neptune's orbit led to the discovery of the object that was causing the perturbation — the planet Pluto. Perturbations in the orbits of stars have led to the discovery of planetary systems outside of our Solar system.

And according to the *Collins Dictionary of Mathematics:*

> **perturbation,** n , **1.** (of an equation or of an optimization problem) a change (usually slight) in the values of some of the underlying parameters, made to obtain the desired solution or to study the stability of a given solution.
> **2.** (*Mechanics*) a small displacement in the orbit of a particle.

I cite these definitions to make it abundantly clear that the perturbations under consideration in our discussion are **small.** The purport which the term "small perturbation" has in our discussion is given by these definitions.

The concept "small perturbation" is used extensively in mathematics and other fields to investigate the behavior of an object in the neighborhood of a given point in the assumed space. Typically, the *size* of the small perturbations concerned is not stipulated a priori. So, certain perturbations that might eventually be considered, can turn out to be quite large, sometimes very large.

The term "small" often indicates that small perturbations are considered first. Larger perturbations are considered only if the investigation of all smaller perturbations failed to achieve the objective of the analysis.

For example, the *Radius of Stability* model is designed to deal with "small" perturbations in that its task is to identify the **smallest** perturbation in the nominal state that destabilizes the system. Metaphorically, we can describe this model as follows:

· A deflated gray balloon is placed at the nominal state.
· The balloon is inflated slowly, keeping its center at the nominal state.
· The balloon turns **blue** when its surface touches an unstable state.
· We stop inflating the balloon when it turns **blue**.

So clearly, depending on the system under consideration, the size of the balloon when it turns blue can be small, large or even huge. But this does not alter the fact that large sizes are considered only if all smaller sizes “failed the test”.

Figure B.1 illustrates this point. It shows a nominal point c, a curve C and three *admissible* perturbations of c, where a perturbation is admissible iff it “moves” c to a point on the curve C. For simplicity assume that the distances are Euclidean.

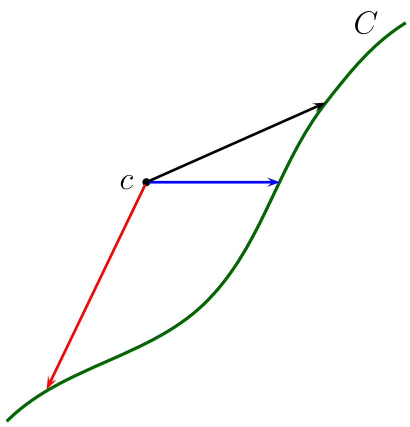


Figure B.1: Three admissible perturbations of c

Observe that none of these perturbations is the smallest admissible perturbation. In fact, the smallest admissible perturbation is shown in Figure B.2. Its size is equal to the *radius* of the smallest circle centered at c that is tangent to the curve C.

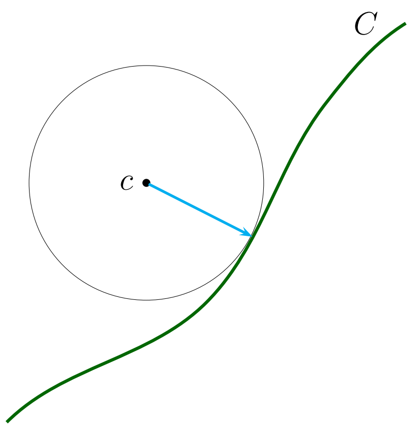


Figure B.2: Smallest admissible perturbation of c

Needless to say, in many applications the numeric value of the “size” of the admissible perturbations depends on the unit used to specify it (e.g. mm, cm, m, km, etc).

In some applications of the *Radius of Stability* model, it is essential to keep the size of the perturbations small because the analysis is valid only in a small neighborhood around the nominal value of the parameter of interest. For instance, this could be the case in situations where the stability conditions concerned are approximations that are valid only in a small neighborhood of the nominal state of the system.

For these reasons the interpretation of the results generated by *Radius of Stability* models must be consistent with the goals and requirements of the application under consideration.

Appendix C

Laplace's Principle of Insufficient Reason

This principle, named after the famous French mathematician and astronomer Pierre-Simon, marquis de Laplace (1749–1827), is also known as the *Principle of Indifference.*

Roughly, the principle argues that if you face $n > 1$ events that are mutually exclusive and collectively exhaustive, then under severe uncertainty, it makes sense to *assume* that these events are *equally likely,* hence that each occurs with probability $1/n$. The most famous applications of this principle are in the exciting area of *gambling*, where all sort of games of luck are played with coins, dice, cards, and wheels.

It also figures prominently in classical decision theory (e.g. Resnik 1987, French 1988) where it is used to model decision-making under severe uncertainty. In this framework, the decision-making situation is described by a *payoff table* where the rows represent the decisions available to the decision-maker (DM) and the columns represent the "states of the world", namely the events that are governed by *Uncertainty,* or *Nature*. The entries in the payoff table represent the awards allotted to the DM, which reflect the decision selected by the DM and the state selected by *Nature*.

For example, consider the *payoff table* shown in Table C.1. The decision maker (DM) has 3 options (decisions) to choose from, and Nature has 5 states to choose from.

		Nature				
		s_1	s_2	s_3	s_4	s_5
DM	a_1	3	2	5	6	2
	a_2	9	8	0	6	7
	a_3	0	5	4	3	0

Table C.1: Payoff table

If the DM selects say, decision a_2, the severity of the uncertainty is manifested in her total lack of knowledge as to which of the five payoffs associated with this decision, namely 9, 8, 0, 6, or 7, will be realized.

In the case of the payoff table given in Table C.1, Laplace's Principle argues that each of the five states occurs with probability $1/5 = 0.2$.

Once this choice of a probabilistic structure is made, the decision-making situation is treated as "decision-making under risk". That is, the decisions are ranked on the basis of the *expected payoff* that they generate.

As a matter of fact, computing the expected values is not really necessary because the sum of the payoffs will yield the same ranking. The expected values are equal to the sums divided by 5 (see Table C.2).

		Nature						
		s_1	s_2	s_3	s_4	s_5	*SUM*	E
DM	a_1	3	2	5	6	2	18	3.6
	a_2	9	8	0	6	7	30	6.0
	a_3	0	5	4	3	0	12	2.4
	$p(s_j)$	0.2	0.2	0.2	0.2	0.2		

Table C.2: Expected Payoffs, $E = \mathit{SUM}/5$

The implication is then that the best decision is a_2. Its expected payoff is equal to 6.

An obvious limitation of the principle is that it cannot be applied in situations where the range of feasible values of the state variable does not submit to a *uniform* probability distribution. For example, if the set of possible values of s is the non-negative section of the real line $\mathbb{R}_+ = [0, \infty)$, then it is impossible to formulate a probabilistic structure such that all the feasible values of the state are equally likely.

One must also be careful with regard to cases where the state is *multivariate* and its components are not "independent". Because, if the assumption in such cases is that a given component is uniformly distributed, the distribution functions of other components may not be uniform. The question is then, how to determine which component should be uniformly distributed. The point is that the answer is not always straightforward.

In fairness to *Laplace's Principle,* it should be noted that this difficulty afflicts distribution functions in general. Needless to say, from a Bayesian point of view, the employment of a *uniform distribution* to quantify severe uncertainty is rather extreme.

C.1 The Size Criterion

It should be noted that in cases where the uncertainty space is *discrete*, Laplace's Principle offers a simple formulation for the *Size Criterion* (section 2.3).

This is done by constructing a *binary* payoff table where 1s represent (decision,state) pairs that are "acceptable" and 0s represent (decision,state) pairs that are "unacceptables". For instance, suppose that in the context of Table C.1, a payoff for decision a_1 is "acceptable" iff it is equal to or greater than 4; a payoff for decision a_2 is "acceptable" iff it is equal to or greater than 8; and a payoff for decision a_3 is "acceptable" iff it is equal to or greater than 2.

Then the binary payoff table representing this situation would be as shown in Table C.3. And the conclusion is that the most robust decision according to the *Size Criterion*, is decision a_3.

It seems that the fact that Laplace's Principle provides a framework for the formulation of the *Size Criterion* is unknown (or not appreciated) in info-gap circles. This is born out, for

		Nature				
	s_1	s_2	s_3	s_4	s_5	*SUM*
DM a_1	0	0	1	1	0	2
DM a_2	1	1	0	0	0	2
DM a_3	0	1	1	1	0	3

Table C.3: Binary payoff table

example, by the attempts to formulate the *Size Criterion* in the context of info-gap robustness models in (Moffitt et al. 2005, 2009, 2010).

Appendix D

More on local vs global robustness

To further illustrate a global approach to robustness, consider the following uncertainty-free optimization problem:

Problem P:

$$\max_{q \in Q} f(q;s) \text{ subject to } r^* \leq r(q;s) \tag{D.1}$$

where f and r are real-valued functions on Q and s is a given parameter.

Now consider a similar problem, except that in this case the value of s is unknown, namely, it is subject to severe uncertainty. Let $S(q)$ denote the set of possible values of s associated with decision $q \in Q$. Given these conditions, we would consider the robust counterpart of Problem P specified by

Problem R:

$$\max_{q \in Q} \min_{s \in S(q)} \{f(q,s) : r^* \leq r(q,s), \forall s \in S(q)\} \tag{D.2}$$

where in this framework f and r have two arguments, namely q and s.

The difficulty is that typically there is no decision $q \in Q$ such that

$$r^* \leq r(q,s) \ , \ \forall s \in S(q) \tag{D.3}$$

Hence, *Problem R* may have no feasible solutions, let alone optimal solutions.

One possible way to get around this difficulty is to relax (D.3) and require instead

$$r^* \leq r(q,s) \ , \ \forall s \in \mathcal{N} \tag{D.4}$$

where $\mathcal{N}$ represents the "normal range" of the state s, or values of s outside this set which can be accepted as (controlled) violations of the constraint, such that the further s is from $\mathcal{N}$, the greater its license to violate the performance constraint $r^* \leq r(q,s)$.

Following this line, we can adopt Ben-Tal et al.'s (2006, 2010) *globalized* approach to robustness, and "relax" the global constraint (D.3) as follows:

$$r^* \leq r(q,s) + \beta \cdot dist(s,\mathcal{N}) \ , \ \forall s \in S(q) \tag{D.5}$$

where $\beta \geq 0$ is a control parameter and $dist(s,\mathcal{N})$ denotes the distance from s to $\mathcal{N}$ based on some suitable metric, such that $dist(s,\mathcal{N}) \geq 0, \forall s \in S(q)$ and $dist(s,\mathcal{N}) = 0, \forall s \in \mathcal{N}$. Note that the relaxed constraint entails (D.4), and for $\beta = 0$ it reverts to the more stringent constraint.

This relaxed global constraint can then be incorporated in a Maximin model as follows:

Problem G:

$$\max_{q \in Q} \min_{s \in S(q)} \{f(q,s) : r^* \leq r(q,s) + \beta \cdot dist(s,\mathcal{N}), \forall s \in S(q)\} \tag{D.6}$$

Note the difference between the three sets of forbidden (unacceptable) $(s, r(q,s))$ values for decision q:

$$\mathcal{F}_{strict} := \{(s,y) : s \in S(q), y < r^*\} \tag{D.7}$$

$$\mathcal{F}_{relaxed} := \{(s,y) : s \in S(q), y < r^* - \beta \cdot dist(s,\mathcal{N})\} \tag{D.8}$$

$$\mathcal{F}_{normal} := \{(s,y) : s \in S(q) \cap \mathcal{N}, y < r^*\} \tag{D.9}$$

In practice, the "normal range" $\mathcal{N}$ can be parameterized by its "size" which can be varied in a sensitivity analysis framework.

The following two examples are designed to highlight the difference between local and global robustness. For simplicity, the examples are presented graphically rather than algebraically. The first features a simple *Radius of Stability* analysis, the second a global robustness analysis based on the discussion in the preceding section.

D.1 Example 1: Radius of stability analysis

Consider a case of three decisions, where $Q = \{q', q'', q'''\}$, and the uncertainty spaces are all equal to the real line $\mathbb{R}$: that is, $S(q') = S(q'') = S(q''') = (-\infty, \infty)$. Also, assume that the respective estimates are all equal to 0, namely $\tilde{s}' = \tilde{s}'' = \tilde{s}''' = 0$.

Let us examine the simple case where, as in info-gap decision theory, the stability regions are determined by a performance requirement $r^* \leq r(q,s)$. The critical performance level r^* is equal to 10, and the performance functions are shown in Figure D.1. Since the uncertainty space is unbounded, only a small section of it, in the neighborhood of the estimate, is shown. Assume that, as shown in the figure, the performance functions retain their trends in both directions.

For simplicity, suppose that the balls in this case are of the form $B(\rho, \tilde{s}) = \{s : |s - \tilde{s}| \leq \rho\}$, whereupon we have $B(\rho, \tilde{s}) = B(\rho, 0) = [-\rho, \rho], \rho \geq 0$.

Since decision q' violates the performance requirement at the estimate $\tilde{s} = 0$, it follows that its *Radius of Stability* is equal to zero: $\rho' = 0$. By inspection, the *Radii of Stability* of q'' and

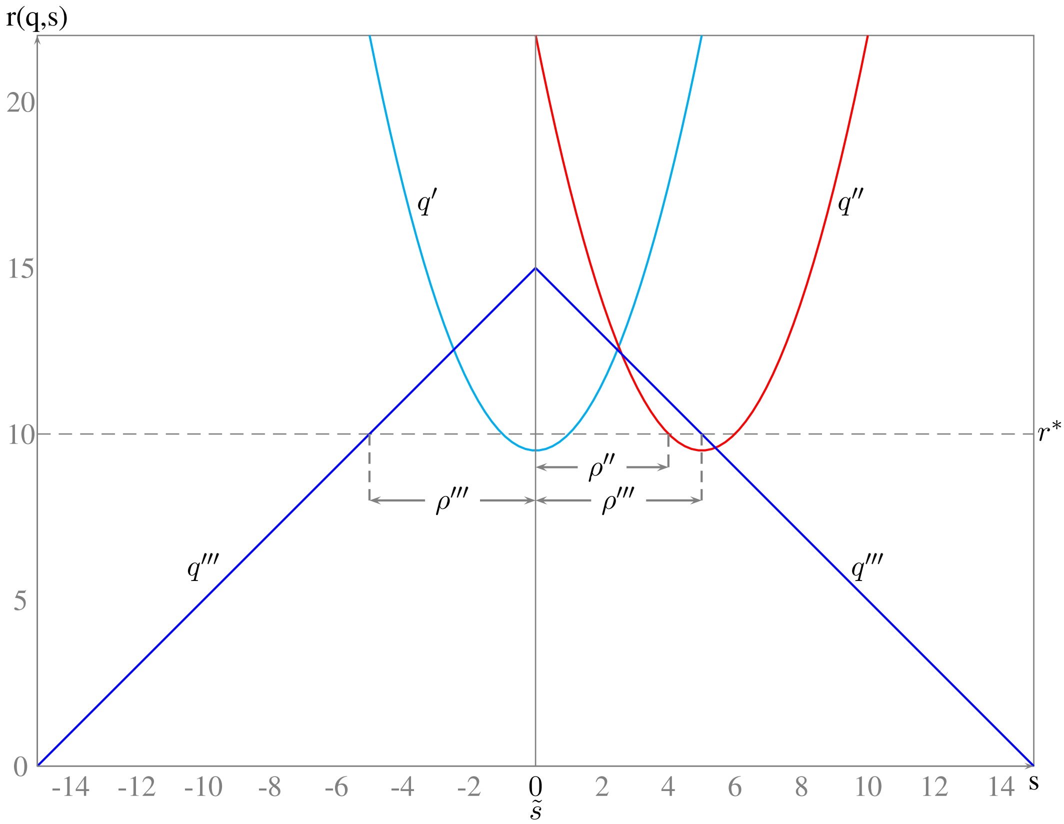


Figure D.1: Radius of stability analysis

q''' are $\rho'' = 4$ and $\rho''' = 5$, respectively.

Hence, according to the *Radius of Stability* approach, the most robust decision at $s = 0$ is q'''. Observe, however, that this decision is manifestly fragile over its uncertainty space $S(q''') = (-\infty, \infty)$.

In contrast, while according to the *Radius of Stability* model, decision q'' is not as robust as q''' at $s = 0$, it is far more robust than q''' globally on the uncertainty space $(-\infty, \infty)$.

This should come as no surprise. The *Radius of Stability* model is not designed to seek global stability/robustness.

D.2 Example 2: A global robustness analysis

Consider an instance of Problem G consisting of 4 decisions:

$$Q = \{q', q'', q''', q''''\}\}$$

$$S(q') = S(q'') = S(q''') = S(q'''') = (-\infty, \infty)$$

$$\tilde{s}' = \tilde{s}'' = \tilde{s}''' = \tilde{s}'''' = 0$$

$$r^* = 10$$

$$\mathcal{N} = [-2, 2]$$

$$\beta = 1$$

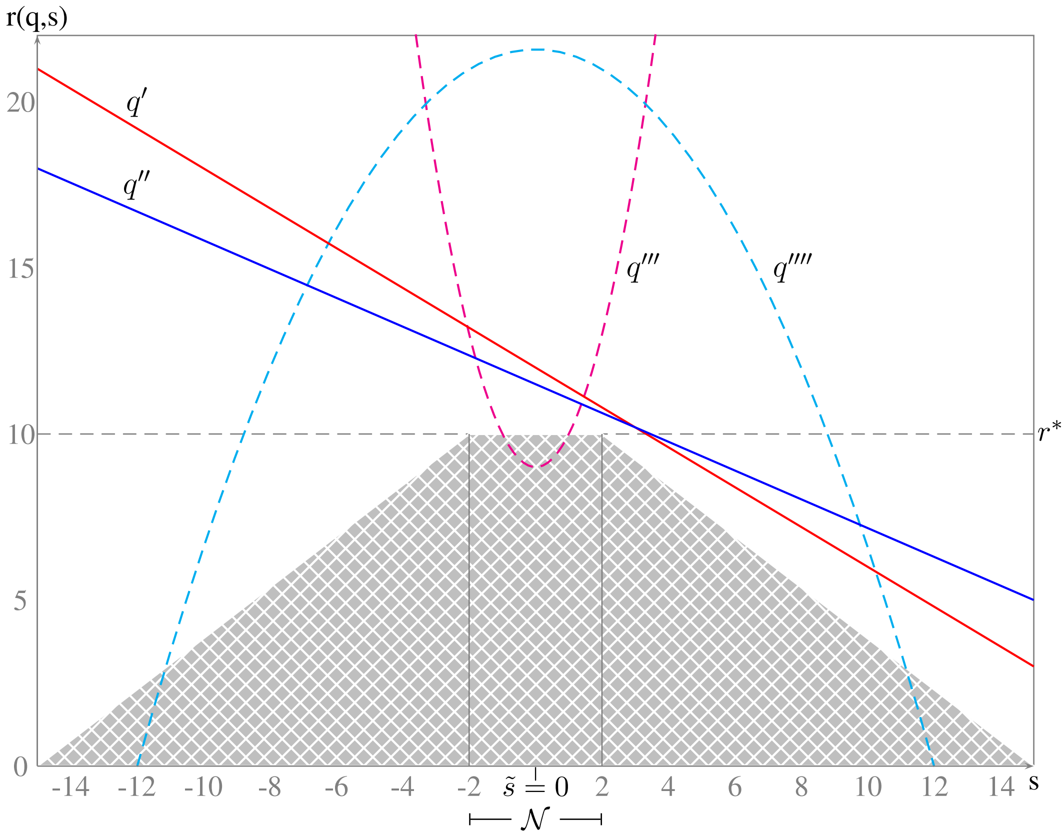


Figure D.2: Relaxed Forbidden region $\mathcal{F}_{relaxed}$

$$dist(s,\mathcal{N}) = \min_{s'\in\mathcal{N}} |s-s'| = \begin{cases} 0 & , \; s \in [-2,2] \\ |s|-2 & , \; s \notin [-2,2] \end{cases}$$

The performance functions are shown in Figure D.2 and the cross hatched area represents the relaxed "forbidden" region, noting that here

$$\mathcal{F}_{strict} = \{(s,y) : s \in (-\infty,\infty), y < 10)\} \tag{D.10}$$

$$\mathcal{F}_{relaxed} = \{(s,y) : s \in (-\infty,\infty), y < 10 - dist(s,\mathcal{N})\} \tag{D.11}$$

$$\mathcal{F}_{normal} = \{(s,y) : s \in [-2,2], y < 10\} \tag{D.12}$$

Observe that according to the *Radius of Stability* model based on these performance functions, the most robust, hence optimal decision, is q'''.

So, by inspection, decisions q''' and q'''' are inadmissible by the "relaxed" constraint: their graphs intrude into the "forbidden" region $\mathcal{F}_{relaxed}$.

To determine which decision is optimal with respect to the globalized approach, we consider the worst values of $f(q',s)$ and $f(q'',s)$ over the uncertainty space $(-\infty,\infty)$, and we select the best worst case. This is shown in Figure D.3.

The worst case of $f(q',s)$ is attained at $s=-7$ for which we have $f(q',-7)=4.5$ and the worst case of $f(q'',s)$ is attained at at $s=0$ for which we have $f(q'',0)=0$. Hence, the best worst case is generated by q' and therefore it is the optimal decision in this case.

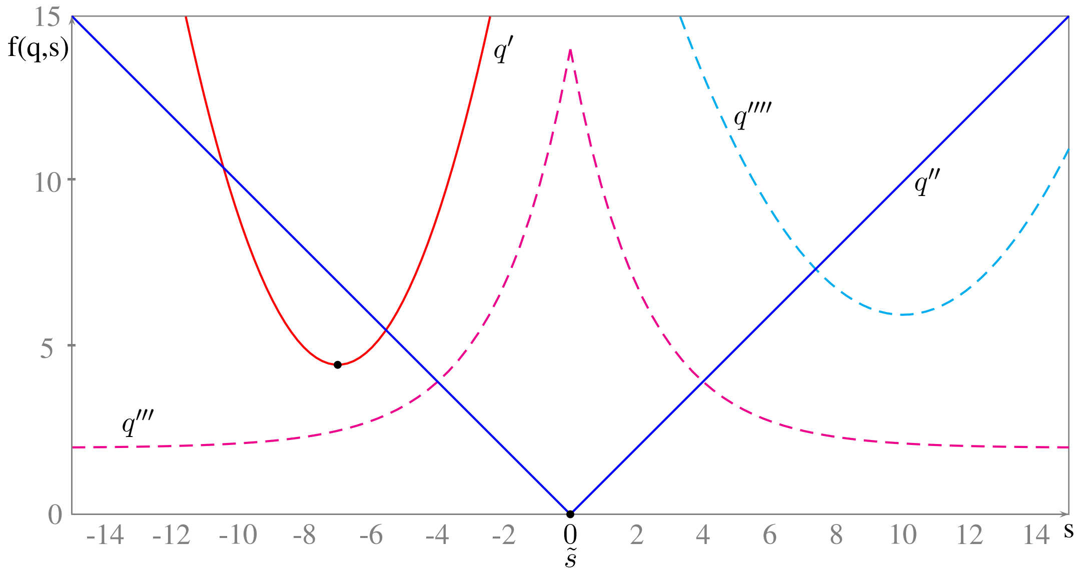


Figure D.3: Objective functions and best worst case

Observe that the worst case of $f(q'''', s)$ is attained at $s = 10$ and is equal to $f(q'''', 10) = 6$, which is better than the worst case of the optimal decision q'. However, as noted above, decision q'''' is inadmissible because it violates the "relaxed" global constraint (D.5).

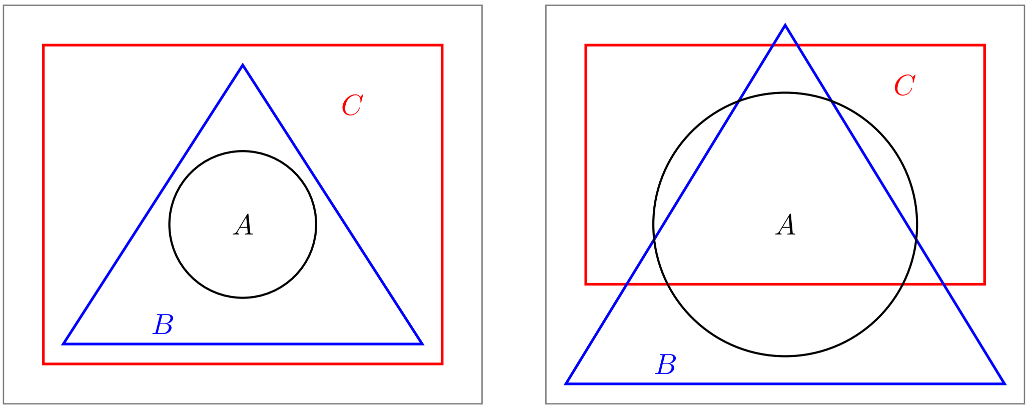
D.3 Nested regions of stability

It should be clear by now that local stability/robustness and global stability/robustness are two entirely different concepts, providing for two fundamentally different types of stability. However, there are cases where the preference of decisions determined by a local stability analysis is similar, even identical, to that determined by a global stability analysis.

For this to be the case in the context of a *Radius of Stability* model, the model must obviously have a property ensuring that its ranking of a system is certain to be very similar to the ranking based on the "size" of a system's region of stability. In this case, this property would ensure that if the *Radius of Stability* of system q' is larger than the *Radius of Stability* of system q'', then the region of stability of system q' is larger than (contained in) the region of stability of system q''. Or, in short, this property would ensure that the ranking of decisions according to their *Radii of Stability* is similar to the ranking generated by the *Size Criterion*.

Now, recall that a *Radius of Stability* model defines robustness as the radius of the largest **ball** (centered at a nominal point) that is contained in the regions of stability. Since by definition balls are **nested**, to ensure that the regions of stability "mimic" the shapes of the balls, it is clearly logical to require that they be nested as well. With this in mind, consider the following:

Definition D.3.1 *Nested sets.*
We say that two sets, say A and B, are NESTED, *iff either $A \subseteq B$, or $B \subseteq A$, or both. A collection of sets is said to be nested iff any two sets in this collection are nested.*



(a) Nested sets: $A \subset B \subset C$ (b) Sets that are not nested

Figure D.4: Illustration of nested sets

This intuitive definition is illustrated in Figure D.4.

In the case of conventional "balls", $B(\rho, c), \rho \geq 0$, centered at a point $c \in \mathcal{B}$, the "radius" ρ is a measure of the size of the set and $\rho' < \rho''$ implies that $B(\rho', c) \subseteq B(\rho'', c)$ (see Appendix A).

Here, the definition of nested sets does not impose such neat properties, which means that nested sets can be much less "structured" than conventional balls. In practice, however, the nesting property is often associated with a parameter that is conceptually similar to the "radius" associated with conventional balls. Hence, the main difference between "balls" and nested sets is that the latter do not necessarily center around an explicit "center point" and that no explicit "distance" function is used in the definition/construction of the content of the set. Still, this definition has merit in the insight that it gives into the relation between local and global robustness models.

Definition D.3.2

A nested Radius of Stability model is a Radius of Stability model whose regions of stability, $S_{stable}(q), q \in Q$, are nested.

An info-gap robustness model whose sets of acceptable values of u, namely $\mathscr{U}(q) := \{u \in \mathscr{U}, r^* \leq r(q, u)\}, q \in Q$, are nested, is a nested info-gap robustness model.

D.3.1 Example

Consider a *Radius of Stability* model whose regions of stability are specified as follows:

$$S_{stable}(q) = \{s \in S(q) : f(s) \leq g(q)\} \ , \ q \in Q \tag{D.13}$$

where f is a real valued function on some set $\mathcal{S}$ and g is a real valued function on Q, with $\mathcal{S}$ being some set such that $S(q) \subseteq \mathcal{S}, \forall q \in Q$.

Observe that, in this case, by inspection:

- $g(q'') = g(q')$ implies $S_{stable}(q'') = S_{stable}(q')$

· $g(q'') < g(q')$ implies $S_{stable}(q'') \subseteq S_{stable}(q')$

Hence, these regions of stability are nested and $g(q)$ can be regarded as a measure of the size of $S_{stable}(q)$.

D.3.2 Example

Consider the case where the sets of acceptable values of u associated with an info-gap robustness model are specified as follows:

$$\mathscr{U}(q) = \{u \in \mathbb{R} : f(q) + ug(q) \leq r^*\} \ , \ q \in Q \tag{D.14}$$

where f and g are real valued function on Q and $g(q) > 0, \forall q \in Q$. We thus have

$$\mathscr{U}(q) = \left\{u \in \mathbb{R} : u \leq \frac{r^* - f(q)}{g(q)}\right\} = (-\infty, h(q)] \ , \ q \in Q \tag{D.15}$$

where

$$h(q) := \frac{r^* - f(q)}{g(q)} \ , \ q \in Q \tag{D.16}$$

Clearly, by inspection,

· $h(q'') = h(q')$ implies $\mathscr{U}(q'') = \mathscr{U}(q')$
· $h(q'') < g(q')$ implies $\mathscr{U}(q'') \subset \mathscr{U}(q')$

Hence, the sets $\mathscr{U}(q), q \in Q$, are nested and $h(q)$ can be regarded as a measure of the "size" of $\mathscr{U}(q)$.

The following definition is inspired by the definition of the *Radius of Stability*:

Definition D.3.3 *Inner radius of a set.*
Let $\mathcal{B}$ be some set on which a neighborhood structure, $B(\rho, c), \rho \geq 0$, around a point $c \in \mathcal{B}$, is defined, and let

$$ir(A) := \max_{\rho \geq 0} \{\rho : B(\rho, c) \subseteq A\} \ , \ A \subseteq \mathcal{B} \tag{D.17}$$

By definition then, $ir(A)$ denotes the radius, ρ, of the largest ball $B(\rho, c)$ that is contained in A. We refer to $ir(A)$ as the INNER RADIUS *of set A.*

So, by definition, the inner radius of a region of stability $S_{stable}(q)$ is the radius of stability of q (at the given center point c). Similarly, the inner radius of the set of acceptable values of u, say $U(q)$, associated with a given estimate $\tilde{u}$, is the info-gap robustness of q.

Note that the value of the inner radius of set A would typically depend on the neighborhood structure under consideration, including the value of the center point c. However, the neighborhood structure under consideration will typically not affect the ranking of sets that is based on their size according to the *set containment criterion.*

Theorem D.3.1 *Let $\mathcal{B}$ be some set on which a neighborhood structure, $B(\rho, c), \rho \geq 0$, around a point $c \in \mathcal{B}$ is defined, and let A' and A'' be two nested subsets of $\mathcal{B}$. Then,*

$$ir(A'') < ir(A') \longrightarrow \quad A'' \subset A' \tag{D.18}$$
$$A'' \subset A' \quad \longrightarrow ir(A'') \leq ir(A') \tag{D.19}$$

Proof. Assume that, under the conditions specified above, $ir(A'') < ir(A')$. Then, it follows from the definition of an inner radius of a set that A'' contains the ball $B(ir(A''), c)$, but it does not follow that it contains the (larger) ball $B(ir(A'), c)$. On the other hand, A' contains both balls. Since the sets A' and A'' are nested, this implies that $A'' \subset A'$.

To prove the second part of the theorem, observe that from the first part of the theorem it follows that if $A'' \subset A'$, then $ir(A'') > ir(A')$ cannot be the case. Hence, $A'' \subset A'$ entails that $ir(A'') \leq ir(A')$. QED

Figure D.5 illustrates the reasoning behind this theorem. Set A' (represented by the red rectangle) contains set A'' (blue triangle). Hence, the inner radius of A' (denoted ρ') is not smaller than the inner radius of A'' (denoted ρ''). Conversely, since the inner radius of A' is strictly larger than the inner radius of set A'', set A'' is a proper subset of A'.

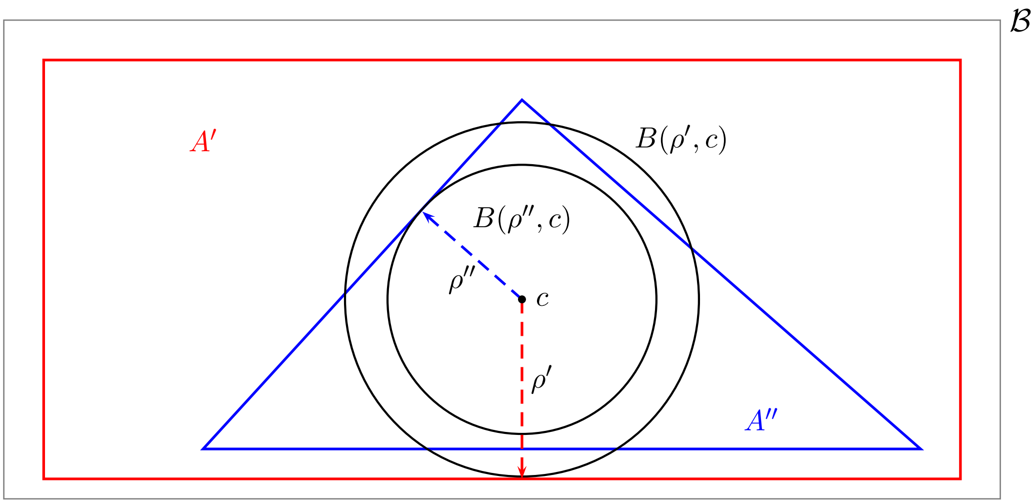


Figure D.5: Reasoning behind Theorem D.3.1: $\rho' = ir(A')$, $\rho'' = ir(A'')$.

The situation depicted in Figure D.6 explains the $\leq$ (rather than $<$) on the right hand side of (D.19). That is, it shows a case where A'' is a proper subset of A', yet $ir(A') = ir(A'')$.

It is of course possible to strengthen the results formulated in Theorem D.3.1 by imposing stricter nesting conditions on the nested sets, so as to obtain the more potent result

$$A'' \subset A' \longleftrightarrow ir(A'')) < ir(A') \tag{D.20}$$

But, I shall not pursue this rather technical matter here.

Of more immediate importance for the *Radius of Stability* model, hence info-gap's robustness model, is the following direct consequence of Theorem D.3.1.

Corollary D.3.1 *Assume that the regions of stability associated with the Radius of Stability*

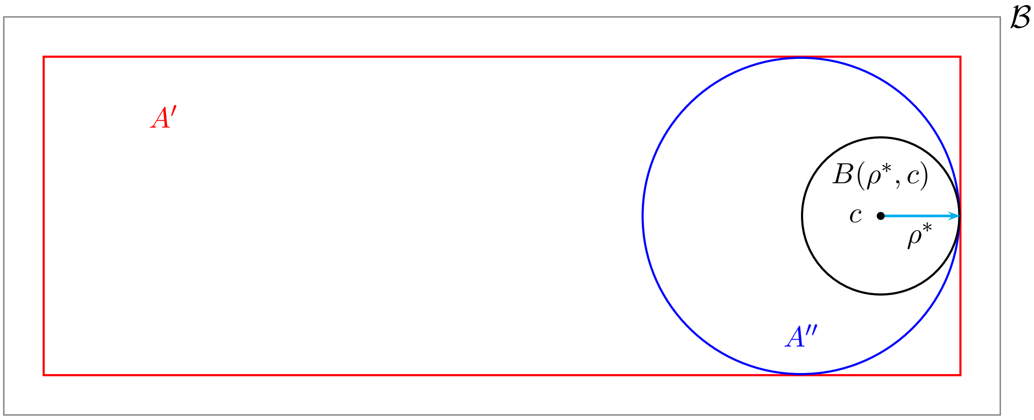


Figure D.6: A case where $A'' \subset A'$, yet $ir(A') = ir(A'') = \rho^*$

model are nested, and let $(q', q'') \in Q^2$ be any pair of systems. Then,

$$\rho(q'', s^*) < \rho(q', s^*) \longrightarrow S_{stable}(q'') \subset S_{stable}(q') \tag{D.21}$$

$$S_{stable}(q'') \subset S_{stable}(q') \longrightarrow \rho(q'', s^*) \leq \rho(q', s^*) \tag{D.22}$$

Similarly, assume that the sets of acceptable values of u associated with info-gap's robustness model are nested, and let $(q', q'') \in Q^2$ be any pair of systems. Then,

$$\hat{\alpha}(q'', \tilde{u}) < \hat{\alpha}(q', \tilde{u}) \longrightarrow \mathscr{U}(q'') \subset \mathscr{U}(q') \tag{D.23}$$

$$\mathscr{U}(q'') \subset \mathscr{U}(q') \longrightarrow \hat{\alpha}(q'', \tilde{u}) \leq \hat{\alpha}(q', \tilde{u}) \tag{D.24}$$

But, it is important to note that imposing nesting conditions such as those required by this corollary usually simplifies the robustness problem under consideration to such an extent that it is rendered *trivial*. So much so that there is clearly no need to use the *Radius of Stability* model for its solution! In other words, such conditions usually produce a simplified robustness problem that can be solved directly by an analysis of the "size" of the regions of stability.

D.3.3 Example

Consider again the regions of stability specified by (D.13), namely

$$S_{stable}(q) = \{s \in S(q) : f(s) \leq g(q)\}, \; q \in Q \tag{D.25}$$

Since the "size" of $S_{stable}(q)$ is non-decreasing with $g(q)$, we can compare the (global) robustness of the systems by comparing their $g(q)$ values. In particular, to select the most robust system we can select the system that maximizes $g(q)$ over $q \in Q$. Thus, the robustness problem in this case simply amounts to $\max_{q \in Q} g(q)$.

But more than this. As the following example illustrates, even patently simple regions of stability do not satisfy the nesting property. The inference therefore is that only extremely simple (degenerate ?) regions of stability have this property.

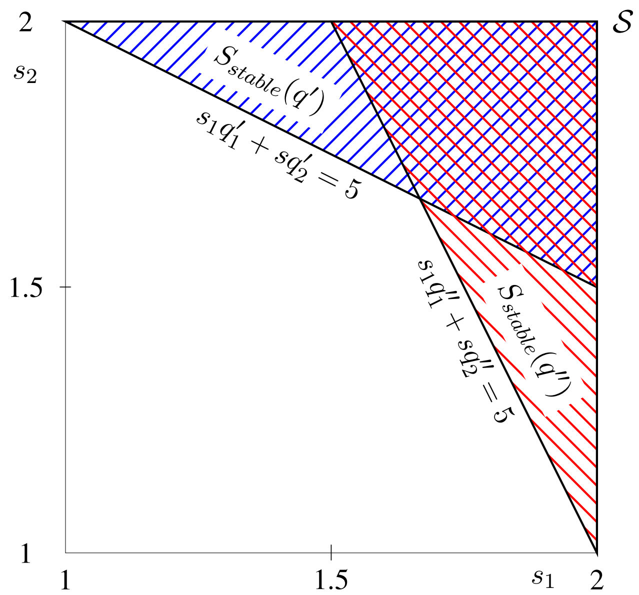


Figure D.7: Two non-nested regions of stability

D.3.4 Example

Consider the regions of stability specified by

$$S_{stable}(q) = \{s \in \mathcal{S} := [1,2]^2 : s_1q_1 + s_2q_2 \geq 5\}\ ,\ q \in Q := [0,\infty)^2 \tag{D.26}$$

The regions of stability associated with $q' = (1,2)$ and $q'' = (2,1)$ are shown in Figure D.7. Clearly, these regions are not nested.

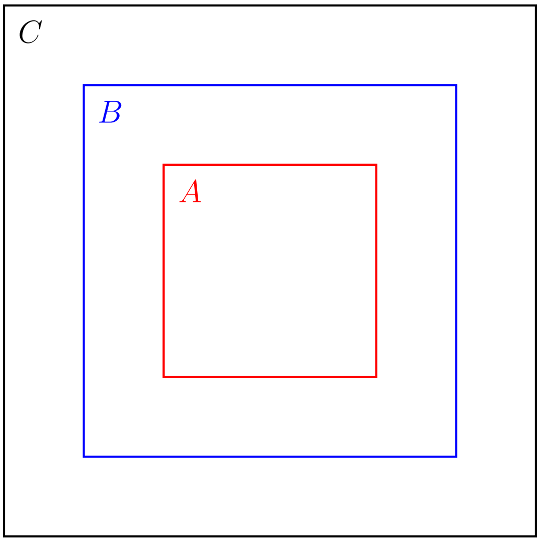
Finally, the following example illustrates why the imposition of the nesting conditions on the *Radius of Stability* model renders the ranking generated by the model invariant with the neighborhood structure of the model.

D.3.5 Example

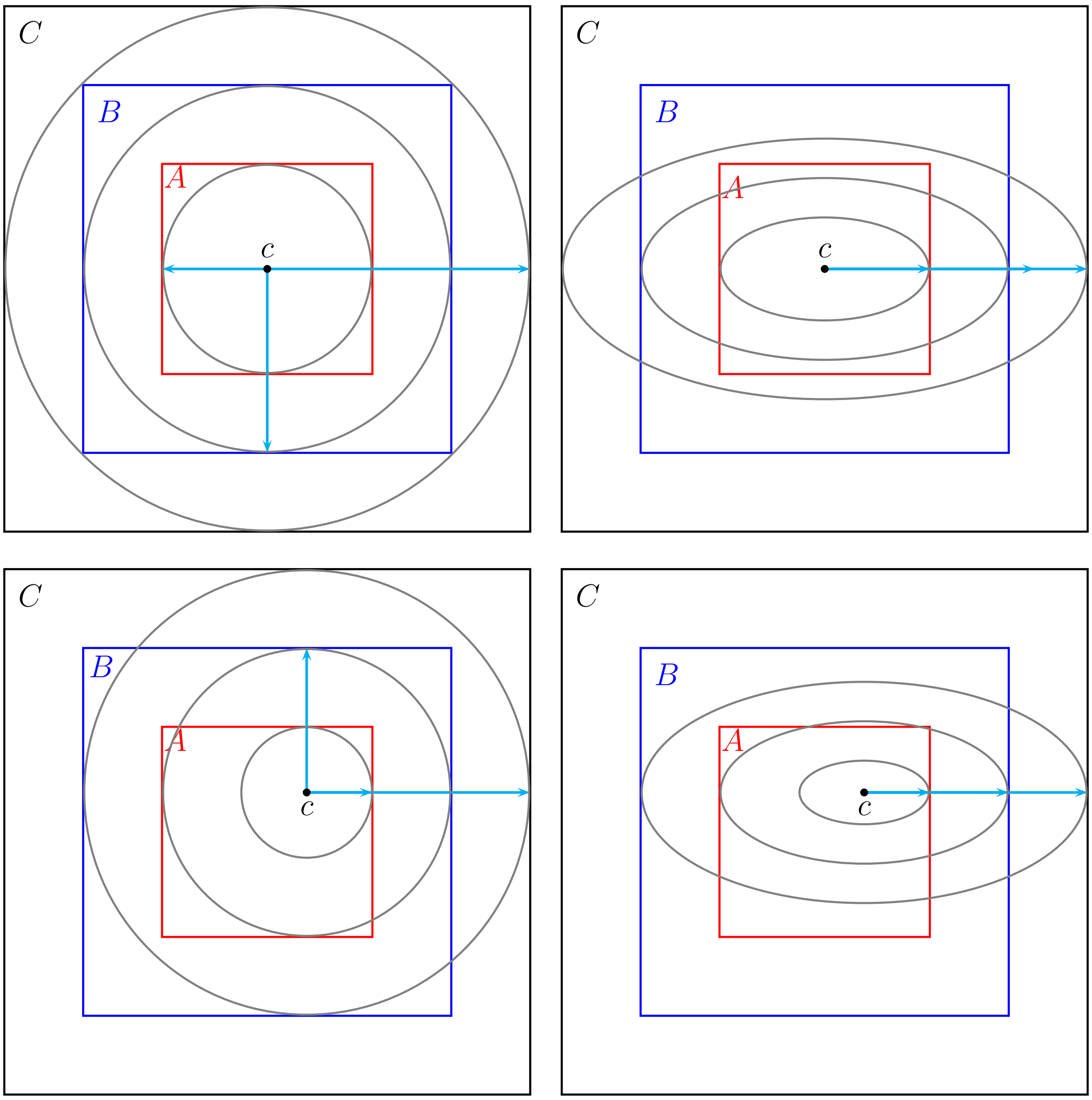
Consider the three nested sets shown in Figure D.8(a). By inspection, we conclude that $ir(A) < ir(B) < ir(C)$ regardless of how the (nested) balls $B(\rho, c), \rho \geq 0$, are defined and where the center point c is located, provided that it is an element of the smallest set, namely A. Note that if the center point c is not a member of a set, then the inner radius of the set is not well-defined.

Figure D.8(b) displays two different neighborhood structures (circles and ellipses), which yield the same ranking based on the inner radii of the sets, namely $ir(A) < ir(B) < ir(C)$. The inner radii associated with two center points are shown so as to emphasize that the ranking is independent of the center point $c \in A$.

Note that if the center point c is not a member of a set, then the inner radius of the set is not well defined.



(a)



(b)

Figure D.8: Three nested sets and their inner radii

D.3.6 Summary

While it is no doubt true that there are cases where models of local robustness can be used to rank decisions according to their global robustness, the use of models of local robustness to determine the global robustness of decisions is highly limited. This is so because:

· Conditions such as those imposed by the nested sets property are highly restrictive.

· In cases where these conditions hold, the robustness problem is typically so simple that its solution clearly does not require the use of formal robustness models such as *Radius of Stability* models.

The discussion on the nested sets continues in Appendix E.

D.4 Bibliographic notes

The distinction between local and global robustness/stability and the importance of drawing this distinction is of course well-known. For instance, consider this:

> It is well known that local and global stability are not equivalent and that it is much easier to test for local stability than for global stability (see LaSalle, 1976). The point of this paper is to show that for the usual one-dimensional population models, local and global stability are equivalent.
>
> Cull (1981, p. 47)

Of particular interest to this discussion is the following statement:

> If a common best critical region does not exist, . . . which minimize the type II error locally, that is to say with respect to alternative hypotheses in the neighborhood of the hypothesis considered. In this paper we develop methods for the determination of a system of regions acceptance taking in account type II errors also relative to alternative hypotheses not lying in the neighborhood of the hypothesis to be tested.
>
> Wald (1939, p. 47)

Its interest is due to the fact that it was made in Wald's (1939) first article on the Maximin model.

The following quote illustrates that it is quite easy to spell out clearly the meaning of local robustness:

> **2. Basic theory** In this section, we describe a general framework for local robustness analysis. By local robustness analysis, we refer to the idea that a policymaker may not know the 'true' model for some outcome of interest, but may have sufficient information to identify a space of potential models that is local to an initial baseline model. This may be regarded as a conservative approach to introducing model uncertainty into policy analysis, in that we start with a standard problem (identification of an optimal policy given a particular economic model) and extend the analysis to a local space of models, one that is defined by proximity to this initial baseline. The local model uncertainty

> assumption, in our judgment, is naturally associated with minimax approaches to policy evaluation. When a model space includes nonlocal alternatives, we would argue that one needs to account for posterior model probabilities in order to avoid implausible models from determining policy choice.
>
> Brock and Durlauf (2005, p. 2070)

Indeed, a clear, edifying description of the *Radius of Stability* model can apparently be given even without the term “local” being stated explicitly. For instance:

> Stability radius of linear normal distributed parameter systems with multiple directional perturbation.
>
> Abstract
>
> In this note, the stability robustness problem of linear time-invariant normal distributed parameter systems with multiple bounded or relative bounded directional perturbations is considered. The Lyapunov stability criterion is used to derive **the system stability radius, i.e. the extent of perturbation within which the system can keep stability.**
>
> Lu and Fong (1998, p. 819, emphasis added)

This abstract conveys the local nature of the stability under consideration without it resorting to the term “local”!

Appendix E

Info-gap proxy theorems

E.1 Introduction

A considerable effort had been devoted in the info-gap literature (e.g Ben-Haim 2006, 2007a, 2009, Ben-Haim and Cogan 2011, Davidovitch 2009) to the formulation of conditions aimed at insuring that info-gap's robustness models act as proxies of "probability of success" models. The idea here is that under these conditions, info-gap's robustness models compute the probability of the performance requirement $r^* \leq r(q,u)$ being satisfied, assuming that $u \in \mathscr{U}$ is the realization of a random variable induced by some unknown probability distribution function on the uncertainty space $\mathscr{U}$.

Theorems specifying such conditions are called in the info-gap literature "proxy theorems". Their object is to enable the ranking of decisions according to their "probability of success" via info-gap's non-probabilistic robustness models.

Now, given that info-gap's robustness model is, by definition, a model of local robustness and "probability of success" models are inherently models of "global robustness", it only stands to reason that, methodologically, such theorems will require the imposition of stringent conditions on info-gap's robustness model. Consequently, one would expect such conditions to greatly simplify the robustness problems (both local and global) under consideration.

The point is that for such theorems to exist, the sets of acceptable values of u, namely $\mathscr{U}(q) := \{u \in \mathscr{U} : r^* \leq r(q,u)\}, q \in Q$, must be "in tune" with info-gap's regions of uncertainty, $U(\alpha, \tilde{u}), \alpha \geq 0$. And for this to be the case, exacting conditions must be imposed on the performance function, r, of the robustness model.

This point was examined and illustrated in detail in section D.3 in the discussion on the relationship between local and global robustness and the notion of *nested sets*. So, as "probability of success" models are inherently global in nature, the discussion in this appendix is in fact a continuation of the discussion in section D.3

E.2 Proxy theorems

The most comprehensive examination of this topic can be found in Davidovitch (2009) where a distinction is made between *strong proxy theorems* and *weak proxy theorems*.

Members of the *strong* class do not impose any requirement on the probability/likelihood structure of the uncertainty space under consideration. So the task of "strong proxy theorems" is to stipulate conditions guaranteeing that the decision selected by info-gap's robustness model maximize the "probability of success", regardless of the probability/likelihood structure.

In contrast, the *weak* class imposes some "coherency" conditions on the probability/likelihood structure. These conditions are designed to ensure that the probability/likelihood structure associated with the uncertainty space "mimic" the implicit "distance" function that is used in info-gap's model of uncertainty to create the neighborhoods $U(\alpha, \tilde{u}), \alpha \geq 0$ around the estimate $\tilde{u}$. Hence, if u' is further from $\tilde{u}$ than u'', then u' is "less likely" than u''.

It is important to take note that Davidovitch's (2009) overall conclusion is that proxy theorems are expected to be "very rare":

> We have shown that the definition of strong proxy theorems discussed by Ben-Haim (2007), is very restrictive, and that when the uncertainty is multi-dimensional, strong proxy theorems are expected to be very rare. Then we shall prove that even this weaker definition does not hold for a wide family of common problems.
>
> Davidovitch (2009, p. 137)

Since the technical issues are discussed in detail in Davidovitch (2009), I shall not elaborate on them here.

However, an issue that is central to this entire enterprise is not even broached in Davidovitch (2009). It must therefore be raised and clarified here. So, harking back to my discussion in section D.3 on the relation between global and local stability/robustness, recall that the conditions imposed by proxy theorems on info-gap's robustness model yield global robustness models that are typically significantly simpler than the associated "proxy" info-gap robustness models.

The question is therefore this:

> Given that the stringent requirements imposed by the proxy theorems have the effect of rendering the **global** robustness problem under consideration **trivially simple,** what is the merit, the point, the advantage, of using info-gap's robustness model for this purpose? More generally, what is the point, the merit, the advantage of such proxy theorems?

The following examples illustrate this point.

E.3 Example

Consider the *Model Mixing* problem examined in Ben-Haim (2007a). Using our notation, the performance function is as follows:

$$r(q,u) = a(q) + ub(q) \ , \ q \in Q, u \in \mathscr{U} = \mathbb{R} \tag{E.1}$$

where a and b are real valued functions on Q, with $B(q) \neq 0, \forall q \in Q$. The performance requirement is $r(q,u) \leq r^*$.

Thus, the "critical" value of u, namely the value of u for which $r^* = r(q,u)$ is equal to

$$u_c(q) := \frac{r^* - a(q)}{b(q)} \ , \ q \in Q \tag{E.2}$$

Therefore, the set of acceptable values of u associated with decision q is as follows:

$$\mathscr{U}(q) := \{u \in \mathbb{R} : r(q,u) \leq r^*\} \ , \ q \in Q \tag{E.3}$$
$$= \{u \in \mathbb{R} : a(q) + ub(q) \leq r^*\} \tag{E.4}$$
$$= \begin{cases} (-\infty, u_c(q)] & , \ b(q) > 0 \\ [u_c(q), \infty) & , \ b(q) < 0 \end{cases} \tag{E.5}$$

Note that if $b(q) > 0$ then $\mathscr{U}(q)$ is increasing in size with $u_c(q)$, hence the global robustness of q is increasing with $u_c(q)$. Similarly, if $b(q) < 0$ then $\mathscr{U}(q)$ is decreasing in size with $u_c(q)$, hence the global robustness of q is decreasing with $u_c(q)$.

Therefore, if $b(q) > 0, \forall q \in Q$, then the most globally robust decision is one that maximizes $u_c(q)$ over $q \in Q$. In this case, the global robustness problem boils down to this:

$$\max_{q \in Q} u_c(q) \equiv \max_{q \in Q} \frac{r^* - a(q)}{b(q)} \tag{E.6}$$

And if $b(q) < 0, \forall q \in Q$, then the most globally robust decision is one that minimizes $u_c(q)$ over $q \in Q$. In this case, the global robustness problem boils down to this:

$$\min_{q \in Q} u_c(q) \equiv \min_{q \in Q} \frac{r^* - a(q)}{b(q)} \tag{E.7}$$

The point is that these conclusions are obtained directly from the performance constraint by *inspection*. Meaning that there is no call for a formal local robustness model such as that proposed by info-gap decision theory.

E.3.1 Example

In a recent publication, Ben-Haim and Cogan (2011, p. 12) considered the case where the requirement constraint is as follows:

$$|X_c(q)| - |X_r(u)| \geq \delta \tag{E.8}$$

where

- $\mathscr{U} \subseteq \mathbb{R}^3$ (uncertainty space).
- $Q \subseteq \mathbb{R}^3$ (decision space).
- $\delta \in \mathbb{R}$ (critical performance level).
- $|X_r(u)|$ = greatest positive and purely real root of a given polynomial whose coefficients depends on $u \in \mathscr{U}$.

· $|X_c(q)|$ = magnitude of the greatest real or complex root of a given polynomial whose coefficients depend on $q \in Q$.

This implies that the "probability of success" is as follows:

$$Prob(|X_c(q)| - |X_r(u)| \geq \delta) = Prob(|X_c(q)| - \delta \geq |X_r(u)|) \tag{E.9}$$
$$= Prob(u \in \mathscr{U}(q)) \tag{E.10}$$

where

$$\mathscr{U}(q) = \{u \in \mathscr{U} : |X_c(q)| - \delta \geq |X_r(u)|\} \ , \ q \in Q \tag{E.11}$$

Thus, **by inspection,** since the size of $\mathscr{U}(q)$ is increasing with $X_c(q)$ and these sets are nested, the "probability of success" is non-decreasing with $|X_c(q)|$. Therefore, the decision that maximizes the "probability of success" is one that maximizes the value $|X_c(q)|$ over $q \in Q$. In other words, the global robustness problem under consideration boils down to this:

$$\max_{q \in Q} |X_c(q)| \tag{E.12}$$

The point is that to reach this conclusions there is no need to consider the much more complicated optimization problem that is specified by info-gap's robustness model, which in this case is as follows:

$$\hat{\alpha}(\tilde{u}) := \max_{q \in Q} \max \left\{ \alpha \geq 0 : \delta \leq \min_{u \in U(\alpha, \tilde{u})} |X_c(q)| - |X_r(u)| \right\} \tag{E.13}$$

Note that the sets of acceptable values of u specified by (E.11), namely $\mathscr{U}(q)$, are NESTED (see section D.3), hence the proxy theorem in Ben-Haim and Cogan (2011, p. 13) is an immediate implication of Theorem D.3.1. But this is hardly surprising. Indeed, the results in Davidovitch (2009, section 6.2, pp. 130-131) are contingent on the sets of acceptable values of u being nested.

Furthermore, since the size of $\mathscr{U}(q)$ is non-decreasing with $|X_c(q)|$, it follows that $|X_c(q)|$ can be used as a measure of the global robustness of decision q.

E.3.2 Example

There are strong proxy theorems (see Ben-Haim 2007a and Davidovitch 2009) for cases where the uncertainty space $\mathscr{U}$ is a subset of the real line $\mathbb{R}$ and for each $q \in Q$ the performance level $r(q, u)$ is monotone (increasing or decreasing) with u. For simplicity assume that $\mathscr{U}$ is bounded, $r(q, u)$ is non-decreasing with u and that the performance constraint is of the form $r^* \leq r(q, u)$. In this case, the sets of acceptable values of u are as follows

$$\mathscr{U}(q) = \{u \in \mathscr{U} : r^* \leq r(q, u)\} \ , \ q \in Q \tag{E.14}$$
$$= [\underline{u}(q), \overline{u}] \tag{E.15}$$

where $\overline{u}$ is the largest element of $\mathscr{U}$ and $\underline{u}(q)$ is the smallest element of $\mathscr{U}$ such that $r^* \leq r(q,u)$.

Since $\overline{u}$ does not depend on q, it follows that the size of $\mathscr{U}(q)$ is decreasing with $\underline{u}(q)$. Hence, the smaller $\underline{u}(q)$, the more robust q. The global robustness problem is then as follows:

$$\min_{q\in Q} \underline{u}(q) \tag{E.16}$$

The robustness problem is so simple that there is no need for a "formal" robustness model to reach this obvious conclusion.

It is interesting to note that the point estimate $\tilde{u}$ which, according to the precepts of info-gap decision theory, is the fulcrum of an info-gap robustness analysis, in all these examples, has no role whatsoever in the global robustness analysis. That is, other than requiring, for convenience sake, that $\tilde{u} \in \mathscr{U}(q), \forall q \in Q$, the value of $\tilde{u}$ has not the slightest impact on the ranking of the decisions, nor on the choice of the optimal decision. The same is true of the regions of uncertainty, $U(\alpha,\tilde{u}), \alpha \geq 0$, which, I need hardly remind the reader, are the backbone of info-gap decision theory's robustness model. In these examples the regions of uncertainty, $U(\alpha,\tilde{u}), \alpha \geq 0$, have no bearing at all on the ranking of the decisions.

In sum, what these simple examples bring out is that the real issue with info-gap "proxy theorems" is not that they are "very rare". The real issue is that the conditions that they require induce such a simplification of the robustness problem under consideration, that the use of an info-gap robustness model as a "proxy" is rendered utterly redundant, in fact counter-productive. A global measure of robustness that is based on the size of the sets of acceptable values of u can be used (directly) instead.

Two other points must be stressed:

1. *The distinction between local and global robustness*
It is surprising that although the analysis of strong proxy theorems in the info-gap literature is centered on the "size" of the sets of acceptable values of u, no awareness is shown of the fact that while info-gap's robustness model is by definition a model of local robustness the "size" criterion is a criterion of global robustness. Consequently, no account is taken of the fact that strong proxy theorems involve conditions designed to ensure that the ranking of decisions based on their local robustness be similar to a ranking based on their global robustness.

2. *Relation to Radius of Stability model*
Global robustness issues associated with *Radius of Stability* models are immediately relevant to the discussion on proxy theorems in the info-gap literature (e.g. see the reference to Cull 1981 in section D.4)

For example, one of the early applications of *Radius of Stability* models were in the context of stability issues related to dynamical systems (see Packard and Doyle 1993, Zhao and Stoustrup 1997, Paice and Wirth 1998, Hinrichsen and Pritchard 2010, and references therein.) Similarly, stability of polynomials has been an extensive area of research for many years now (e.g. Hinrichsen and Pritchard 1992, 2010, Tolstobrov 1997).

Consider, for instance, the following quote, consisting of the abstract of the paper: *Computation of stability radius for polynomials* by Graillat and Langlois (2004, p. 1):

> A polynomial is stable if all its roots have negative real part, and unstable otherwise. For a stable polynomial, the distance to the nearest unstable polynomial is an important parameter in control theory for example. In this paper, we focus on this distance called the stability radius of polynomial p. We propose to modify the level contour function of the pseudozero set to derive a bisection algorithm that computes an arbitrary accurate approximation of this stability radius. Numerical simulations and comparisons with pseudozero graphics are here after presented.
>
> **Key words:** pseudozero, abscissa mapping, stability radius, robust stability, bisection.

Clearly, these investigations are immediately pertinent to articles in the info-gap literature, such as Ben-Haim and Cogan (2011).

It is surprising therefore that there are no references in the info-gap literature to models of local robustness of the *Radius of Stability* type.

Appendix F

Correcting the flaws

There are various ways to amend info-gap decision theory, depending on the intended use of its robustness model. I briefly examine two approaches to this task:

· Keeping info-gap's robustness model intact.

· Modifying info-gap's robustness model.

I shall refer to these approaches as *Radius of Stability correction* and *Structural correction*, respectively.

F.1 Radius of Stability correction

This approach is straightforward. All one needs to do in this case is to simply call a spade a spade. That is, treat info-gap's robustness model as the model it is: a simple *Radius of Stability* model. In practical terms this would mean that the accepted info-gap language would have to be made consistent with that pertaining to *Radius of Stability* models.

The most important ingredient of this change in orientation would have to reflect the fact that in the framework of the *Radius of Stability* model, the robustness of a decision is the size of the **largest acceptable worst-case perturbation** in the given nominal value of the parameter of interest. The **worst-case** clause indicates that for each given "size", all perturbations not exceeding the given size are considered. This is illustrated in Figure F.1.

It shows two balls centered at the estimate $\tilde{u}$. All the perturbations within the small ball satisfy the performance requirement $r^* \leq r(q,u)$, hence the worst-case perturbation within this ball is acceptable. On the other hand, in the case of the large ball, not all the perturbations within this ball satisfy the performance requirement. Hence, the worst-case perturbation does not satisfy the performance requirement in this case, and consequently the worst-case perturbation within this ball is unacceptable.

Note that in the context of info-gap's robustness model there are only two "cases", namely

· Case A: the perturbation in $\tilde{u}$ under consideration *violates* the performance constraint, hence it is *unacceptable.*

· Case B: the perturbation in $\tilde{u}$ under consideration *satisfies* the performance constraint, hence it is *acceptable.*

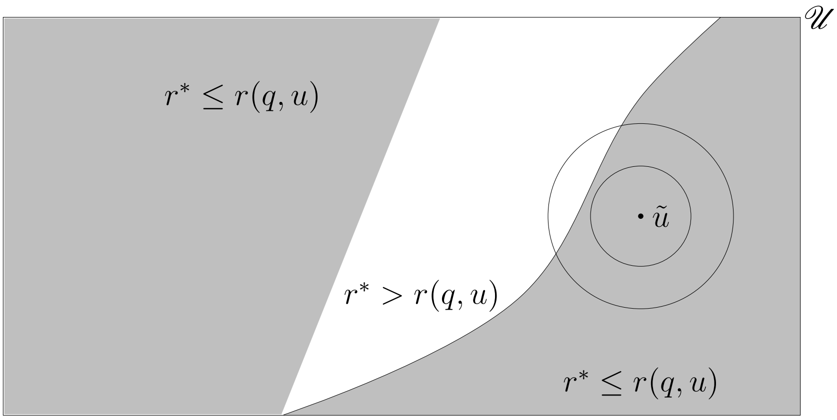


Figure F.1: Worst-case perturbations

In other words, the "case" refers to the performance constraint $r^* \leq r(q,u)$, not to the value of the performance function $r(q,u)$ associated with a pair $(q,u) \in Q \times \mathscr{U}$. Since there are only two cases, for every $\alpha \geq 0$, there is a "worst case" of u in $U(\alpha,\tilde{u})$. Furthermore, there is a "worst case" of u in $\mathscr{U}$.

Note that the concept "uncertainty" is not even mentioned in the above discussion. This is so for the obvious reason that the *Radius of Stability* model is designed for the modeling and analysis of *perturbations* rather than uncertainty as such. Furthermore, the robustness analysis is local, which means that it is unsuitable for the modeling and analysis of *global* robustness against severe uncertainty of the type described in the info-gap literature.

The advantage of this approach is self-evident: it gives access to a mainstream model of robustness. Indeed, should this approach be adopted, the reference to info-gap decision theory would be made redundant.

F.2 Structural correction

It is, of course, possible to introduce structural changes in info-gap's robustness model with the view to transform it from a *Radius of Stability* model to a model of *robustness against severe uncertainty.* It should be noted, however, that these changes will not be cosmetic. Indeed, they will change info-gap decision theory's character altogether.

For example, it is rather easy to impose a likelihood structure on info-gap's uncertainty space and then appeal to this structure to define global robustness over the entire uncertainty space.

But in this case, why turn to info-gap decision theory in the first place? After all, info-gap decision theory *per se* does not offer any insight into the formulation of such likelihood structures.

Alternatively, suppose that we allow info-gap's uncertainty model to retain its non-probabilistic and likelihood-free characteristics, but we change the definition of robustness to render it *global*. Again, info-gap decision theory *per se* does not offer any special guidance as to the measures required to accomplish such a reformulation.

Note that one simple (obvious) modification is to replace the neighborhood structure specified by the nested balls $U(\alpha, \tilde{u}), \alpha \geq 0$, by arbitrary *subsets* of $\mathscr{U}$. The associated robustness model would then be as follows:

$$\hat{\alpha}(q) := \max_{V \subseteq \mathscr{U}} \{size(V) : r^* \leq r(q, u), \forall u \in V\} \ , \ q \in Q \tag{F.1}$$

This, however, is the robustness model associated with the *Size Criterion* (see section 2.3).

In short, while one can envisage structural changes in info-gap decision theory that will transform its model of local robustness into a model of robustness against severe uncertainty, the inspiration for such structural changes will have to come from external sources (theories).

What this means then is that before we set out to implement a structural correction of info-gap decision theory whose aim is to transform its model of local robustness into a model of robustness against severe uncertainty, it would make more sense to turn to existing models of global robustness that already do the job.

The rich literature on robust decision-making in general and robust optimization in particular is an obvious starting point.

Appendix G

Ben-Haim's (2007) FAQs

In this appendix I take up the seven FAQs addressed in Ben-Haim (2007), but not the answers given therein. Each question has a short and a long version. The short versions are as follows:

1. Does an Info-Gap Model only Deal with Local Uncertainty?
2. Are Info-Gap Models of Uncertainty Based on the Principle of Ignorance?
3. Is Robust-Satisficing the Same as Max-Min?
4. Can the Max-Min Strategy be Used to Describe Robust-Satisficing Behavior?
5. Does Maximum Robustness Imply Maximum Likelihood of Success?
6. Can Max-Min Computational Tools be Used for Info-Gap Robustness?
7. Can Info-Gap Theory Deal with Multiple Performance Requirements?

I now address each question, and its long version. In each case I quote both versions.

G.1 Local nature of info-gap decision theory

> Does an Info-Gap Model only Deal with Local Uncertainty?
>
> **Question:** The best estimate, $\tilde{u}$, of an info-gap model of uncertainty is sometimes a wild guess, since in most cases the horizon of uncertainty, α, is unknown. How sure can we be that an info-gap model of uncertainty $U(\alpha, \tilde{u})$ is not just a local analysis of risks which grossly errs in the true value u? Is it not preferable to employ qualitative methods for managing "unknown-unknowns"? Does the info-gap approach simply sweep major risks under the carpet?
>
> Ben-Haim (2007, p. 2)

My Answer:
The fact is that by virtue of being a *Radius of Stability* model, info-gap's robustness model is a model of *local* robustness par excellence. As we saw above, this means that, by definition, (except for trivial cases where robustness is not an issue!) it takes no account of the bulk of the uncertainty space, the implication being that it ignores the potentially large *No Man's Land*

hence the severity of the uncertainty. The inference therefore is that info-gap decision theory does indeed sweep major risks under the carpet. The picture (see Figure G.1) leaves no room for debate on this matter.

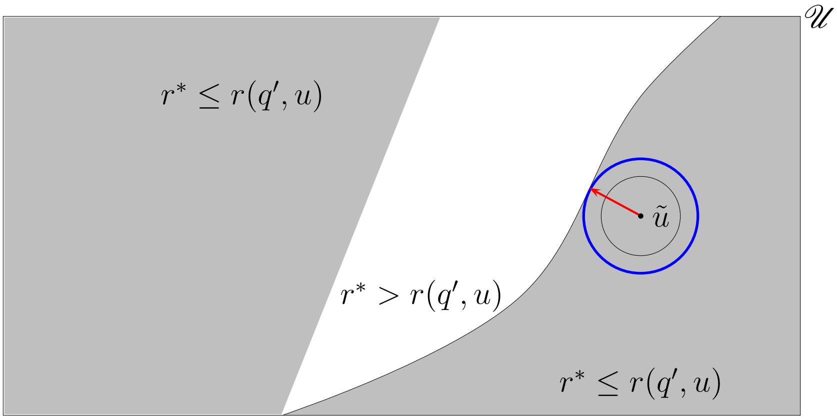


Figure G.1: Info-gap's robustness of q' at $\tilde{u}$

The sub-question *Is it not preferable to employ qualitative methods for managing "unknown-unknowns"?* is intriguing. It suggests, albeit implicitly, that the choice one has here is between "info-gap decision theory" and "qualitative" methods. It thus gives the grossly misleading impression that info-gap decision theory is the only non-probabilistic "quantitative" theory available for dealing with decision-making under severe uncertainty.

G.2 Laplace's Principle of Insufficient Reason

Are Info-Gap Models of Uncertainty Based on the Principle of Ignorance?

> **Question:** Are info-gap models of uncertainty based on the principle of ignorance (also known as the principle of insufficient reason, or maximum entropy)? Do info-gap models implicitly assume a uniform probability distribution?
>
> Ben-Haim (2007, p. 4)

Info-gap models of uncertainty are not based on *Laplace's Principle of Insufficient Reason.* Info-gap's robustness (hence, uncertainty) model is based on a local worst-case approach to uncertainty. That is, the underlying assumption here is that *Nature* (namely Uncertainty) always selects the most undesirable (worst) outcome vis-a-vis the decision maker). So, in the case of info-gap's robustness model, if the decision maker selects decision q and a horizon of uncertainty α, *Nature* (Uncertainty) will select the worst u in the ball $U(\alpha, \tilde{u})$, insofar as the performance requirement $r^* \leq r(q, u)$ is concerned. For instance, She may select the value of u in the ball $U(\alpha, \tilde{u})$ that minimizes the value of $r(q, u)$.

G.3 The Maximin connection

Is Robust-Satisficing the Same as Max-Min?

> **Question:** Is info-gap theory simply a re-invention of the max-min principle? Does info-gap robust-satisficing lead to the same decisions as a max-min decision strategy?
>
> Ben-Haim (2007, p. 5)

My Answer:
Info-gap theory is not “simply a re-invention of the max-min principle", simply because info-gap decision theory’s robust-satisficing model is not equivalent to Wald’s Maximin paradigm. The latter is incomparably more general and powerful. The exact relationship between info-gap’s robust-satisficing and Wald’s Maximin paradign is as follows:

- Info-gap’s robustness model and info-gap’s decision model are simple instances of Wald’s Maximin model.
- Info-gap’s robustness model and info-gap’s decision model cannot be the same as Wald’s Maximin model simply because Wald’s Maximin model is the general case (the prototype) and info-gap’s robustness and decision models are specific cases of this general case.
- This means that, not only methodologically but also practically, the capabilities and scope of Wald’s Maximin paradigm are incomparably more general and powerful than those of info-gap’s robustness model and info-gap’s decision model.
- That simple instance of Wald’s Maximin model that is equivalent to info-gap’s robustness model is known universally as “radius of stability” model.
- Thus, info-gap’s robustness model is not a re-invention of Wald’s Maximin model, it is in fact a re-invention of the *Radius of Stability* model.
- That instance of Wald’s Maximin model that is equivalent to info-gap’s decision model generates exactly the same decisions as those generated by info-gap’s decision model.

It should be stressed that the Maximin representation of info-gap’s robustness model is simple and straightforward both conceptually and mathematically:

$$\underbrace{\max\ \{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\}}_{\text{Info-gap's robustness model}} \ \equiv \ \underbrace{\max_{\alpha\geq 0}\ \min_{u\in U(\alpha,\tilde{u})} f(q,\alpha,u)}_{\text{Maximin representation}} \tag{G.1}$$

where

$$f(q,\alpha,u) := \begin{cases} \alpha & , \ r^* \leq r(q,u) \\ -\infty & , \ r^* > r(q,u) \end{cases} \tag{G.2}$$

Similarly, for info-gap’s decision model we have:

$$\underbrace{\max_{q\in Q} \max\{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\}}_{\text{Info-gap's decision model}} \ \equiv \ \underbrace{\max_{q\in Q,\alpha\geq 0}\ \min_{u\in U(\alpha,\tilde{u})} f(q,\alpha,u)}_{\text{Maximin representation}} \tag{G.3}$$

Remark:
It is possible to express the relationship between f and the performance constraint $r^* \leq r(q,u)$ more explicitly.

For instance, observe that if $r^* \leq r(q,\tilde{u})$, then

$$f(q,\alpha,u) = \alpha \cdot (r^* \diamond r(q,u)) \tag{G.4}$$

where the binary function $\diamond$ is defined as follows:

$$a \diamond b := \begin{cases} 1 & , \ a \leq b \\ 0 & , \ a > b \end{cases} \tag{G.5}$$

We thus have,

$$\underbrace{\max\left\{\alpha \geq 0 : r^* \leq r(q,u), \forall u \in U(\alpha,\tilde{u})\right\}}_{\text{Info-gap's robustness model}} \equiv \underbrace{\max_{\alpha \geq 0} \min_{u \in U(\alpha,\tilde{u})} \alpha \diamond (r^* \leq r(q,u))}_{\text{Maximin representation}} \tag{G.6}$$

This formulation of the Maximin representation of info-gap's robustness model highlights the fact that robustness is sought with respect to the constraint $r^* \leq r(q,u)$.

G.4 More on the Maximin connection

> Can the Max-Min Strategy be Used to Describe Robust-Satisficing Behavior?
>
> **Question:** Can the max-min strategy always be used to describe robust-satisficing behavior?
>
> Ben-Haim (2007, p. 8)

My Answer:
As indicated above, since info-gap's robustness model is an instance of Wald's Maximin model, a (proper) Maximin representation of info-gap's robustness model always yields exactly the same results as those generated by info-gap's robustness model. Since the correspondence between info-gap's robustness model and its Maximin representation is direct and transparent, the Maximin representation of info-gap's robustness model can always be used to describe the behavior of info-gap's robustness model.

G.5 Info-gap robustness and likelihood of success

> Does Maximum Robustness Imply Maximum Likelihood of Success
>
> **Question:** Does maximum robustness imply maximum likelihood of success? Even though an info- gap model of uncertainty is non-probabilistic, does the info-gap robustness function nonetheless reveal something about the underlying probability? Ben-Haim (2007, p. 9)

My Answer:
Given that info-gap decision theory is non-probabilistic and likelihood-free, in general, its robustness function does not reveal anything about a possible "latent" underlying probabilistic structure that quantifies the uncertainty associated with the true value of the parameter of interest.

To establish a formal connection between such a "latent" probabilistic structure and info-gap's robustness model, it is imperative to impose exacting conditions on the relation between info-gap's robustness model and this "latent" probabilistic structure. But there are several methodological and technical reasons to believe that formulating such conditions is well-nigh impossible, except for some very simple (trivial) cases (see Appendix E).

That is, it is possible to formulate an info-gap robustness model that is subject to certain (highly restrictive) conditions, so that it will mimic the behavior of probabilistic models. But this is limited only to special, for the most part trivial, cases. As indicated by Davidovitch (2009, p. 137):

> We have shown that the definition of strong proxy theorems discussed by Ben-Haim (2007), is very restrictive, and that when the uncertainty is multi-dimensional, strong proxy theorems are expected to be very rare. Then we shall prove that even this weaker definition does not hold for a wide family of common problems.

Furthermore, the restrictive assumptions that are required to establish the connection between info-gap's robustness model and the "latent" probabilistic structure are not applicable in cases where the uncertainty is severe. So what is the point of this entire exercise?

G.6 Use of Maximin computational tools

> Can Max-Min Computational Tools be Used for Info-Gap Robustness?
>
> **Question:** A lot of effort in statistics goes into finding methods for determining max-min strategies and estimators. Can these tools be used for calculating info-gap robustness functions?
>
> Ben-Haim (2007, p. 10)

My Answer:
Since info-gap's robustness model is a simple instance of Wald's Maximin model, any general purpose computational tool available for Maximin problems would be applicable for the solution of the optimization problems induced by info-gap's robustness model.

The reference to "statistics" in the question is intriguing considering that Maximin models are used extensively in other fields such as control theory, economics and robust optimization.

Furthermore, given that info-gap's robustness model is a Maximin model of the **radius of stability** type, it is clear that the general purpose computational tools that are used in the solution of *Radius of Stability* problems would be used in the solution of the optimization problems yielded by info-gap's robustness model.

G.7 Multiple performance requirements

Can Info-Gap Theory Deal with Multiple Performance Requirements?

Question: Can info-gap theory deal with multiple performance requirements, such as multiple requirements like eq.(1) on p. 9?

Ben-Haim (2007, p. 11)

My Answer: Recall that info-gap's generic robustness model is as follows:

$$\hat{\alpha}(q, \tilde{u}) := \max \{\alpha \geq 0 : r^* \leq r(q, u), \forall u \in U(\alpha, \tilde{u})\} \ , \ q \in Q \tag{G.7}$$

where $r^* \leq r(q, u)$ represents the performance requirement. Formally, in this formulation r is a real valued function on $Q, \times \mathscr{U}$ where $\mathscr{U}$ denotes the uncertainty space.

Thus, as many additional constraints as are required can be imposed on the $(q, u) \in Q \times \mathscr{U}$ pairs. Furthermore, the constraints are not required to be of the $\leq$ or $\geq$ type. For example,

$$\hat{\alpha}(q, \tilde{u}) := \max \{\alpha \geq 0 : r^* \leq r(q, u), \underline{r} \leq h(q, u) \leq \overline{r}, \forall u \in U(\alpha, \tilde{u})\} \ , \ q \in Q \tag{G.8}$$

is a valid info-gap robustness model, where h is a real valued function on $Q \times \mathscr{U}$ and $\underline{r}$ and $\overline{r}$ are given numeric scalars.

However, the incorporation of additional constraints in the model may complicate the associated optimization problem, especially in cases where the optimization problem is solved analytically.

To make a more general point. From the standpoint of the formulation of the generic *Radius of Stability* model, namely:

$$\rho(q, \tilde{u}) := \max \{\alpha \geq 0 : u \in A(q), \forall u \in U(\alpha, \tilde{u})\} \ , \ q \in Q \tag{G.9}$$

it is utterly immaterial what number and types of constraints are used to specify the sets of acceptable values of u associated with decision q, denoted here by $A(q)$. This formulation merely indicates that each decision can have its own set of acceptable values of u. So, insofar as the model is concerned, it is immaterial how these sets are defined/specified.

But, the structure of $A(q)$ is crucial for the solution of the optimization problem induced by the *Radius of Stability* model. In this vein the structure of $r(q, u)$ is crucial for the solution of the optimization problem induced by info-gap's robustness model.

The choice between (G.7) and (G.9) is a matter of style rather than substance. Indeed, these two models are equivalent (see formal proof in Sniedovich, 2010, 2011).

Appendix H

Sniedovich's (2008a) FAQs

Although the basic facts about info-gap decision theory are straightforward and crystal clear, it seems that some facts are not entirely clear to all info-gap users and analysts. Consequently, several unfounded claims and statements about info-gap decision theory including outright misrepresentations of it, have found their way into info-gap publications, even peer-reviewed articles. So, in this section I examine a number of points that seem to require further clarification. A more extensive list can be found in Sniedovich (2008a)

H.1 Is info-gap decision theory unique and radically different from all other theories for decision under uncertainty?

The general impression one gets from the info-gap literature is that info-gap's robustness model is not only unique in the sense that it is a distinct model in its own right, but that it is radically different from mainstream models of uncertainty and robustness.

These misconceptions stem from a lack of familiarity with the relevant literatures of decision theory, control theory, economics, operations research, robust optimization, etc. Thus, some info-gap scholars maintain that info-gap's robustness model is not a Maximin model, others maintain that it is similar but that its mode of operation is different from that of the Maximin model, while others maintain that it is equivalent to Wald's Maximin model. And some even claim that info-gap robustness analysis generalizes Wald's Maximin strategy (See Appendix J).

All these claims are erroneous.

For, as indicated in this manuscript, info-gap's robustness model is a simple *Radius of Stability* model (circa 1960) — itself a simple instance (specific case) of Wald's famous Maximin model. This means, of course, that info-gap's model is neither distinct, nor new, nor radically different from main-stream models of uncertainty and robustness.

H.2 In what sense is info-gap's robustness analysis a worst-case analysis?

It is persistently claimed in the info-gap literature that info-gap's robustness analysis is not a worst-case analysis. This claim has various manifestations: info-gap's analysis is not a worst-case analysis, or info-gap's robustness analysis does not identify a worst case, or there is no worst-case in info-gap's model of uncertainty, etc.

The assertion that there is no worst case, or there cannot be a worst-case for info-gap's horizon of uncertainty, is apparently due to a misapprehension of the implications that an unbounded horizon of uncertainty, α, has for info-gap's robustness analysis. The argument advanced for this purpose is as follows: The fact that the horizon of uncertainty, α, is often unbounded, means that points in the uncertainty space can be arbitrarily distant from the estimate $\tilde{u}$. This means that for every value of u there is a worse value of u, the inference thus being that there is no worst case.

But the point to note is that the preference that info-gap's robustness analysis seeks to establish is not with regard to the values of u per se. Therefore, the worst-case analysis that it conducts is not based on the value of u as such. Rather, the worst-case analysis conducted by info-gap decision theory is with respect to the performance requirement $r^* \leq r(q,u)$. Thus, as indicated in §F.1, insofar as this requirement is concerned, there are only two cases. Hence, in non-trivial situations, there always is a worst case and there always is a best case.

Furthermore, the worst-case analysis that is conducted by info-gap's robustness model is of a *local* nature. For each value of α, the worst-case analysis is conducted on the region of uncertainty $U(\alpha,\tilde{u})$, not over the uncertainty space $\mathscr{U}$. Hence, if $r^* \leq r(q,u)$ for all $u \in U(\alpha,\tilde{u})$ then every $u \in U(\alpha,\tilde{u})$ is a worst case and a best case, otherwise any $u \in U(\alpha,\tilde{u})$ such that $r^* > r(q,u)$ is a worst case, and any $u \in U(\alpha,\tilde{u})$ such that $r^* \leq r(q,u)$ is a best case.

H.3 In what sense is info-gap's robustness model an instance of Wald's Maximin model?

In the accepted sense of the concept "instance".

That is, one can set up the constructs constituting Wald's generic Maximin model in such a way that the resulting model will be equivalent to info-gap's generic robustness model in that both will have identical structures and consequently will yield the same results.[1]

Differently put, Wald's generic Maximin model is a general, all-embracing pliable modeling paradigm that allows the modeler to phrase its constituent constructs in various ways so as to obtain a large array of special cases (instances). The *Radius of Stability* model and info-gap's robustness model are instances obtained from such a formulation. There are of course numerous other interesting instances.

[1] In the same sense that the polynomial $q(x) = 2 + 3x + dx^2$ is an instance of $q(x) = a + bx + cx^2$ where a, b, c and d are real parameters. The instance is specified by $a = 2; b = 3; c = d$.

The immediate implication of this is that there is nothing that info-gap's robustness model can do that Wald's Maximin model cannot do. In contrast, there are many things that Wald's Maximin model can do but info-gap's robustness model cannot do.

H.4 What is the significance of the fact that info-gap's decision model is not a Maximin model of the reward $r(q,u)$?

The fact that info-gap's decision model is not a Maximin model of the reward $r(q,u)$ is of no significance. The only reason that it is taken up here is to point out that this fact is often misconstrued as "proof" that info-gap's decision model is not a Maximin model. Statements to this effect range from a blunt denial that info-gap's robustness model is a Maximin model, to claims that although info-gap's decision model is in some way similar to the Maximin model, it "behaves" differently from the Maximin model.

The argument often used to "prove" such claims is as follows: unless certain conditions are satisfied — see below — there is no assurance that the optimal decision generated by info-gap's decision model is optimal with respect to the following Maximin model:

Maximin model of reward for a give $\alpha^\circ \geq 0$

$$R(\alpha^\circ, \tilde{u}) := \max_{q\in Q} \min_{u\in U(\alpha^\circ,\tilde{u})} r(q,u) \tag{H.1}$$

For example, for $\alpha^\circ > \hat{\alpha}(\tilde{u})$, this Maximin model has no feasible solutions! And for $\alpha^\circ = 0$, this model can yield decisions that are very fragile! On the other hand, for $\alpha^\circ = \hat{\alpha}(\tilde{u})$, all the optimal decisions for this Maximin model are also optimal for info-gap's decision model, observing that $R(\hat{\alpha}(\tilde{u})) \geq r^*$.

Clearly, then, this Maximin model is not equivalent to info-gap's robustness model.

But the point to note in this regard is that this "argument" does not prove that info-gap's robustness model and info-gap's decision models are not Maximin models. All it proves is that info-gap's decision model is not equivalent to the above instance of Wald's maximin model[2].

Indeed, as shown in Chapter 5, constructing Maximin models that are equivalent to info-gap's robustness model and info-gap's decision model is straightforward. This means of course that info-gap's robustness model and info-gap decision models are instances of Wald's generic Maximin model.

In short, it is important to appreciate the real implications of the obvious fact that info-gap's decision model is not a Maximin model of the reward $r(q,u)$ because of the misrepresentations of this fact in the info-gap literature.

[2]For the same reason that showing that x^2+3 is not equivalent to say, x^2+2x+3, does not prove that x^2+3 is not a polynomial of degree 2.

H.5 Does info-gap's robustness model explores the entire uncertainty space ?

No, it does not.

Although it is crystal clear that info-gap decision theory conducts a *local* robustness analysis, for no apparent reason some info-gap scholars claim that this analysis *explores the entire uncertainty space under consideration.*

Recall, however, that when evaluating the robustness of decision q, info-gap's robustness analysis **explores only the neighborhood** $U(\alpha^*, \tilde{u})$ where α^* is the robustness of decision q.

Indeed, as indicated by the *Invariance Property,* info-gap's robustness analysis takes no notice whatsoever of the performance of decision q in areas of the uncertainty space that are outside $U(\alpha^*, \tilde{u})$. This, of course, is a type of analysis that does not even attempt to explore the entire uncertainty space under consideration. And this is precisely what renders info-gap decision theory utterly unsuitable for the treatment of severe uncertainty, especially in cases where the uncertainty space is vast.

H.6 Is info-gap decision theory particularly suitable for cases where the uncertainty space is unbounded?

No, it is not.

To the contrary. Because info-gap's robustness model is a model of **local** robustness, info-gap decision theory is in fact ill-equipped for the task. It does not have the wherewithal required for the modeling, analysis and solution of problems where the uncertainty space is unbounded.

H.7 Is info-gap decision theory suitable for the treatment of "Unknown Unknowns" and "Black Swans"?

Of course not!

After all, "Unknown Unknowns"[3] and "Black Swans"[4] are incomparably more difficult to deal with than "ordinary" rare events. The discussions in §3.8 and some of the other FAQs are relevant here.

[3] See http://en.wikipedia.org/wiki/Unknown_unknowns

[4] See http://en.wikipedia.org/wiki/The_Black_Swan_(Taleb_book)

H.8 Does info-gap's robustness model select a decision that maximizes the likelihood of satisficing the performance requirement?

No, it does not.

Although info-gap decision theory is non-probabilistic and likelihood-free, some info-gap scholars contend that it maximizes the likelihood/chance/reliability of the performance requirement being satisfied.

The truth of course is that info-gap's robustness analysis does not do this. What it does maximize is the radius of the ball all of whose elements satisfy the performance requirement.

It is important to note — as Ben-Haim himself (2001, 2006) takes great pains to stress — that no likelihood/chance/reliability whatsoever can be attributed to any of the results or "events" associated with info-gap's uncertainty models.

H.9 Does info-gap robustness reveal something about the underlying probability of "success"?

No, it does not.

See the discussion on this topic in Appendix E.

Of course, under certain (highly restrictive) conditions it is possible to formulate an info-gap robustness model that will mimic the behavior of probabilistic models. But this is limited only to special, for the most part trivial, cases. In the info-gap literature these relations are expressed as "proxy theorems".

Consider then the following:

> We have shown that the definition of strong proxy theorems discussed by Ben-Haim (2007), is very restrictive, and that when the uncertainty is multi-dimensional, strong proxy theorems are expected to be very rare. Then we shall prove that even this weaker definition does not hold for a wide family of commom problems.
>
> Davidovitch (2009, p. 137)

Suffice it to say that there is a long list of reasons for (reasonably) assuming that, when the uncertainty is severe, such proxy theorems will be very rare, furthermore, they should be expected to be valid only in trivial cases.

H.10 Does info-gap's robustness analysis determine how wrong our model can be without violating the performance requirement?

Strictly speaking, this phrasing is incorrect because it is ambiguous.

It is persistently claimed in the info-gap literature that info-gap's robustness model answers questions such as this:

> How wrong can we be in our estimate before the decision we select leads to an unacceptable outcome?

To see "how wrong" this phrasing is, keep in mind that in the framework of info-gap's robustness model, the true value of the parameter u is unknown, as this value is subject to severe uncertainty. This means that we do not/cannot know "how wrong we are in our estimate" (presumably pre-analysis) let alone "how wrong can we be" (presumably post-analysis). We cannot answer this question because there is no way to answer it.

To illustrate, consider Figure H.1 where the region of "acceptable" values of u is represented by the gray area. Also assume that the distances between elements of $\mathscr{U}$ are Euclidean. The question is: how wrong can $\tilde{u}$ be before it exits the "acceptable" region of u?

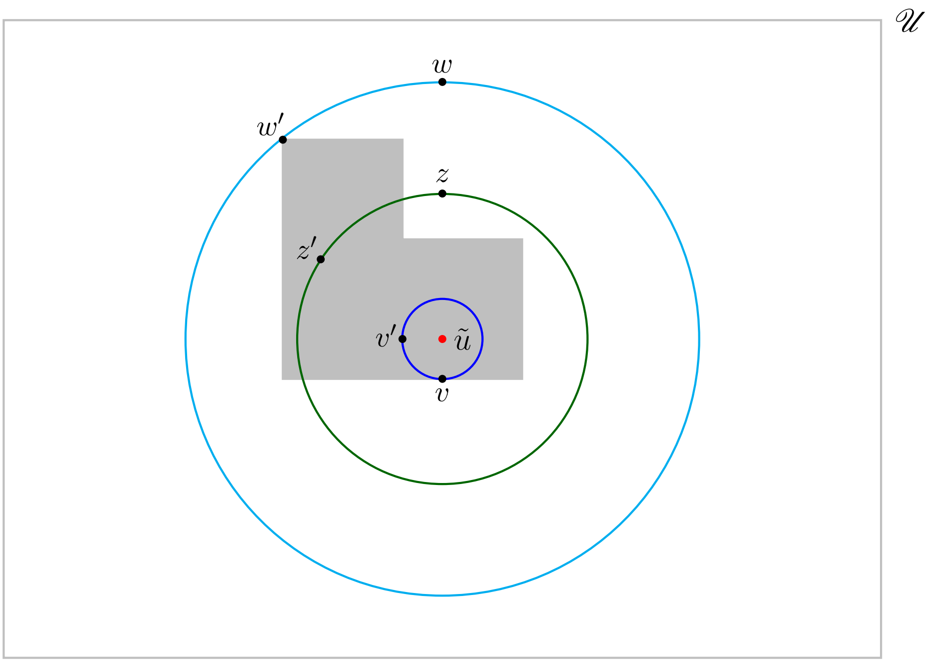


Figure H.1: How wrong can the model (estimate $\tilde{u}$) be?

As brought out by this illustration, this question is ill-defined because the answer may depend on the ... "direction" of the deviation/perturbation in the value of $\tilde{u}$. This is demonstrated by three circles centered at $\tilde{u}$ on each of which two points are marked.

Consider the smallest (blue) circle and the two points on it namely, v and v'. Note that a deviation from $\tilde{u}$ is possible in the neighborhood of v', but not in the neighborhood of v.

And how about the middle (green) circle, and the points z and z'? Note that although both points are at the same distance from $\tilde{u}$, one point (z') is in the acceptable region, whereas the other is not. The same is true with regard to the two points on the large (cyan) circle, namely w and w'.

We can distinguish between two extreme cases:

- If you are an *optimist* (assuming that the deviations are in the "best direction") you'll argue that $\tilde{u}$ can deviate as far as point w' without exiting the acceptable region.

- If you are a *pessimist* (assuming that the deviations are in the "worst direction") you'll no doubt argue that point v is the critical point.

In the framework of info-gap's robustness model, the phrase "how wrong" in questions such as the one posed above, can make sense, only if it is appended by the clause "under the worst-case". In this case we would say that, in the framework of info-gap's robustness model, the robustness of decision q answers the following question:

> How wrong can the estimate $\tilde{u}$ be — in the **worst case** sense — before the true value of u violates the performance requirement?

Note that the "worst case" clause refers to a specific "direction" of the deviation/perturbation. Larger deviations/perturbations in other "directions" may still be safe before the true value of u violates the performance requirement.

The above distinction between the *worst* and *best* further illustrates the fact that info-gap robustness analysis is a (local) worst-case analysis par excellence.

H.11 What exactly is the significance of the Invariance Property of *Radius of Stability* models?

The significance of this property is in the warning signals that it sends about the consequences of (mis)applying *Radius of Stability* models to the pursuit of global robustness. Specifically, it illustrates that these models can misrepresent the robustness of decisions that are locally robust (in the neighborhood of the estimate) as globally robust over the entire parameter space.

In a nutshell, this property makes vivid that — methodologically — *Radius of Stability* models, hence info-gap's robustness model, are utterly unsuitable for the treatment of severe uncertainty of the type addressed by info-gap decision theory.

H.12 Does robust-satisficing have an advantage on optimization of utility?

This is a non-issue.

There is a widespread view in the info-gap literature that in the context of decision-making under severe uncertainty, robust-satisficing is preferable to optimization of utility.

The point is, however, that because any satisficing problem can be easily transformed into an equivalent optimization problem, it follows that the satisficing vs optimizing issue is a matter of style rather than substance. The question that is important in this context is determining WHAT should be satisficed and WHAT should be optimized.

The irony, of course, is that info-gap's robustness model itself is an *optimization* model. The objective of the problem that it defines is to *maximize* the admissible horizon of uncertainty α. In other words, in the case of the info-gap decision model, the utility under consideration is (local) "robustness", hence the optimization model optimizes (maximizes) robustness (utility) subject to a performance requirement.

H.12.1 Remark

In view of the numerous statements in the info-gap literature on this issue, it is interesting to note the following.

According to the Dictionary of Cybernetics (Krippendorff 1986, p. 67, emphasis added):

> **SATISFICING.** By evaluating all possible alternatives, the computation of an optimum strategy may not be feasible when the number of alternatives is very large. E.g., in chess, the number of available plays exceeds computational limits not just for humans. A decision maker who settles for a less ambitious result and obtains the **optimum he can compute under given time or resource constraints is said to satisfice.**

Also note the following quote from the book: *Making Robust Decisions: Decision management for Technical, Business, & Service Teams* (Ullman 2006, pp. 26-27):

> **Decision-Making Truths**
>
> What were you taught about decision making in school? Probably not much. It's likely that the thing you learned was that if you got an answer that matched the one in the book, you got the right one. Yet you later learned that when you went to buy a car, to design a product, or to make a business decision, there where no right answers. In real life:
>
> There are no *right* decisions.
> There are only *satisfactory* decisions.
> Your goal is to find the best possible
> satisfactory decision

The "message" is clear, so that no further elaboration is required: the goal is to find the **best** possible satisfactory decision.

But, reading the info-gap literature, with its repeated hammering on "more is less", "better is better than best", and "satificing performance is superior to maximizing utility", one wouldn't have a clue that there is such a thing called "constrained optimization". It is important, therefore, to draw info-gaps scholars' attention to Chapter 3 *Constrained Optimization* in the book: *Rational Choice* (Gilboa 2010). The opening paragraph reads as follows:

> **3.1 General Framework**
>
> The discussion in chapter 1 concluded that rational choice should distinguish between the desirable and the feasible. Chapter 2 established that *desirable* means "having a higher utility" for an appropriately chosen utility function. Coupling these two ideas, we are led to a model of rational choice as *constrained optimization*, namely choosing an alternative that maximizes utility (or an objective function or a payoff function) given

constraints.

Gilboa (2010, p. 25)

As noted above, in the case of info-gap's decision model, "utility" is the robustness of decisions and info-gap's decision model maximizes this utility subject to a constraint.

H.13 Does info-gap's decision model seek decisions whose sets of acceptable outcomes are the largest?

Of course not. How can it?

Info-gap's decision model seeks decisions whose *Radii of Stability* at a given estimate are the largest. This type of search is, as a matter of principle, different from the search for decisions whose sets of acceptable outcomes are the largest. The *Size Criterion* seeks decisions of this type.

There are, of course, trivial cases where a decision with the largest *Radius of Stability* is also a decision with the largest set of acceptable outcomes. But this is definitely not the general case.

Appendix I

The CSIRO report

I.1 Introduction

At the end of June 2011, after I completed this document, it was brought to my attention that my criticism of info-gap decision theory is referred to, and discussed briefly, in a CSIRO report (henceforth CSIRO REPORT) entitled *Uncertainty and Uncertainty Analysis Methods*, written by Dr. Keith Hayes for ACERA.

The short discussion in this appendix gives a rough sketch of my immediate response to the info-gap content in the CSIRO REPORT and its relation to my document. A detailed discussion on the info-gap content in the CSIRO REPORT can be found on my website at http://info-gap.moshe-online.com/csiro.html.

I.2 Significance

The publication of the CSIRO REPORT is clearly a significant development for my campaign to contain the spread of info-gap decision theory in Australia, but in the first place, it is a significant development for science in Australia (notably applied ecology and environmental management). For, although info-gap decision theory is not the primary concern of the CSIRO REPORT, it does identify and clarify some of the flaws in info-gap decision theory.

This means that, for the first time since I launched my campaign at the end of 2006, scientists in an Australian government organization have stated, in print, views that support my criticism of info-gap decision theory and challenge the validity of claims made in the info-gap literature. It is gratifying, therefore, to read views supporting what I have been arguing publicly for so long, stated on the pages of such a report.

I am afraid, though, that readers will have to read ... between the lines of the CSIRO REPORT to get the full measure of the flaws of the theory that for so long has captivated risk analysts in Australia. For one thing, the CSIRO REPORT does not spell out the obvious conclusions deriving from its analysis of info-gap decision theory — conclusions that are spelled out clearly and unambiguously in my document.

For instance, consider this (emphasis added):

> Analysts who were attracted to IGT because they are very uncertain, and hence reluctant to specify a probability distribution for a model's parameters, may be **disappointed** to find that they need to specify the plausibility of possible parameter values in order to identify a robust management strategy.
>
> Hayes (2011, p. 88)

While this is no doubt the case, it should be pointed out that this "disappointment" is only one of many that are in store for analysts setting out to use info-gap decision theory. These disappointments are the immediate consequence of the fundamental flaw afflicting info-gap decision theory. This is its prescription to apply a model of *local* robustness for the purpose of dealing with a severe uncertainty that is characterized by a vast uncertainty space, a poor point estimate, and a likelihood-free quantification of uncertainty.

The trouble is, however, as attested by the info-gap literature, that info-gap scholars are clearly not disappointed at all by all that, which can only mean that info-gap scholars are unaware of the flaw identified in the CSIRO REPORT, hence of the need to supplement the info-gap methodology with a likelihood/plausibility model.

As a matter of fact, as indicated in my document (see Section 3.11), Hall and Harvey (2009) and Hine and Hall (2010) even go so far as to (mistakenly) assume that such a likelihood/plausibility model is already posited by info-gap decision theory!!!!

In any case, I hope that the discussion on info-gap decision theory in the CSIRO REPORT will encourage other scientists in Australia to speak their minds openly on this matter so as to impress on applied ecologists in the *Land of the Black Swan* how flawed info-gap decision theory actually is.

This is long overdue.

But it is even more important to impress on info-gap scholars and users that there are other theories available for decision-making under severe uncertainty — theories which, **unlike info-gap decision theory**, are suitable for this purpose because they do have the capabilities to perform that which a theory for severe uncertainty ought to be able to do! The point is then that these theories have the capabilities that **info-gap decision theory clearly does not have** — a fact which therefore renders it utterly unsuitable for the treatment of severe uncertainty. As pointed out in my report, the field of *robust optimization* offers a rich literature on this subject.

I.3 Scope

Info-gap decision theory is, on the testimony of its founder, a ... **decision theory.** Its overriding objective is: to identify the best decision under conditions of severe uncertainty. Therefore, in my document I assess it as a ... **decision theory.**

In contrast, the focus in the CSIRO REPORT is on the classification and quantification of uncertainty, and on methods for its analysis, not on *decision-making* as such. Therefore, at first glance, it may appear somewhat surprising that a decision theory is included in Haynes' (2011) investigation at all, and that the chosen odd-fellow is ... info-gap decision theory.

But this only *prima facie* odd choice reflects the fact that info-gap decision theory is extremely popular among applied ecologists in Australia so that it warrants a proper assessment.

What is important is that Hayes (2011) does discuss, clarify, and illustrate one of the major flaws in info-gap decision theory that is discussed in Sniedovich (2007, 2010, 2011). This, as the CSIRO REPORT observes, is that info-gap decision theory does not provide a mechanism to deal with the poor quality of the estimate of the parameter of interest, while at the same time, it focuses its entire *local* robustness analysis in the neighborhood of this estimate. The reason that info-gap decision theory lacks this mechanism is due to the fact that the theory does not posit any likelihood structure in its uncertainty model.

The major part of the assessment of info-gap decision theory in the CSIRO REPORT reads as follows:

> In biosecurity risk assessment one of the most severe forms of uncertainty is our limited understanding of complex ecological processes that manifests as model structure uncertainty. IGT does not provide a ready-made solution to this problem and, as with many other applications of uncertainty analysis, this form of uncertainty is typically not addressed in ecological applications. IGT provides an alternative non-probabilistic way to express uncertainty, but in most ecological applications it is applied to uncertain parameters of probabilistic models, such as the rate of a Poisson process, or the probability of detecting a pest in a trap. Its greatest strength is that it places uncertainty at the forefront of the decision selection problem.
>
> An important point is that its recommendations could be sensitive to the initial estimates of the uncertain parameters. As a method of uncertainty analysis it is not unique in this regard, but, as Figure 4.11 demonstrates, small departures from an initial estimate can still lead to different conclusions.
>
> This is important because IGT does not distinguish between the likelihood of different initial estimates. Hence, if recommendations based on an Info-gap analysis change with different initial estimates, and these estimates are highly uncertain (for example two equally credible experts have different views on the 'best' initial estimate) then the theory may not be able to unambiguously identify the best course of action. If the robustness is low at the point where the preference order of the two decisions change (conditional on the required reward) then the theory highlights that the current level of understanding and information is insufficient for reliable decision-making. This insight, of course, presumes that analysts test for the effect of different initial conditions when using IGT.
>
> Hayes (2011, p. 92)

A careful reading of this assessment reveals that the conclusion that ought to be drawn is that *info-gap decision theory* does not even begin to address the difficulties associated with the fact that the uncertainty under consideration is **severe.**

And I should add that the CSIRO REPORT does not discuss at all the implications of info-gap's robustness model being a simple instance of the *Radius of Stability* model and of Wald's

Maximin model. In fact, there are no references in the CSIRO REPORT to the literature on the treatment of severe uncertainty in decision theory and robust optimization.

I.4 The debate

This leads me to comment on the following statement in the CSIRO REPORT (emphasis added):

> There is, however, an on-going **debate** surrounding IGT that revolves around two claims: a) IGT is not a radically new theory but rather a reformulation of minimax analysis that has been known in the mathematical research literature for over 60 years; and b) its results are sensitive to initial estimates and are not therefore robust to "severe uncertainty" (Sniedovich, 2007, 2008, 2010).
>
> Hayes (2011, p. 88)

As far as I am concerned, there is no "debate" about these two issues. And if it appears that such a debate is on-going, then ... this is a pity, because I do not see that there is anything that is "debatable" here.

This document makes it crystal clear that I proved long ago that info-gap's robustness model is a simple instance of Wald's famous Maximin model (circa 1940), and I am unaware of any argument showing that my proofs are invalid. So what exactly would the debate be about?

Similarly, I proved long ago that info-gap's robustness model is a model of **local** robustness. The proof implies that info-gap decision theory, **by definition**, does not indeed, cannot seek decisions that are robust against severe uncertainty. Rather, all that info-gap decision theory can **by definition** do is seek decisions that are robust against small perturbations in a nominal value of a parameter of interest. And in this regard as well, I am unaware of any argument showing that my proofs are invalid. So what exactly would the debate be about?

Indeed, as I show in this document, the info-gap literature does not debate my formal proofs on this or any other issues.

It is therefore important that Hayes (2011) explains and illustrates the validity of my criticism of info-gap decision theory with regard to point b). It is a pity that this was not done with respect to point a) as well.

As indicated above, a detailed discussion on the info-gap content of the CSIRO REPORT can be found on my website at http://info-gap.moshe-online.com/csiro.html.

Appendix J

The latest news

The November 2011 issue of *Decision Point*[1] refers to the following recently published article on info-gap decision theory:

> Brendan A. Wintle, Sarah A. Bekessy, David A. Keith, Brian W. van Wilgen, Mar Cabeza, Boris Schroder, Silvia B. Carvalho, Alessandra Falcucci, Luigi Maiorano, Tracey J. Regan, Carlo Rondinini, Luigi Boitani and Hugh P. Possingham. Ecological-economic optimization of biodiversity conservation under climate change. *Nature Climate Change,* Volume 1, October 2011, 355-359.

In due course I shall post a detailed review of this article on my website[2]. But, for the purposes of this discussion it suffices to point out the following.

Just as Wintle et al. (2010) heralded a major breakthrough in decision theory with their proposition of an info-gap strategy to tackle *Black Swans* and *Unknown Unknowns,* Wintle et al. (2011) report on a remarkable advance in decision theory (emphasis added):

> Info-gap **generalizes** the maximin strategy by identifying worst-case outcomes at increasing levels (horizons) of uncertainty. This **permits** the construction of 'robustness curves' that describe the decay in guaranteed minimum performance (or worst-case outcome) as uncertainty increases.
>
> Wintle et al. (2011, p. 357)

To appreciate why this statement proclaims a remarkable advance in decision theory, keep in mind that Wald's Maximin model has, for at least five decades, figured as the foremost tool for the treatment of severe uncertainty in areas such as decision theory, robust optimization, and others. The implication is then that a **generalization** of this tool that enables the generation of 'robustness curves' would most certainly count as a major contribution to the state of the art in decision theory.

The fact of the matter is of course, given my discussion in this document, that the claim that "Info-gap **generalizes** the maximin strategy" is nothing short of absurd. And, to show why this is so I need not even work out a detailed argument[3]. Still, it is important to call the

[1] See http://ceed.edu.au/wp-content/uploads/2011/11/DPoint_55.pdf

[2] See http://info-gap.moshe-online.com/reviews/review_33.html

[3] If A is an instance of B then A cannot possibly generalize B.

reader's attention to the fact that claims of this nature attest to a profound misunderstanding of the relationship between info-gap decision theory and Wald's Maximin paradigm.

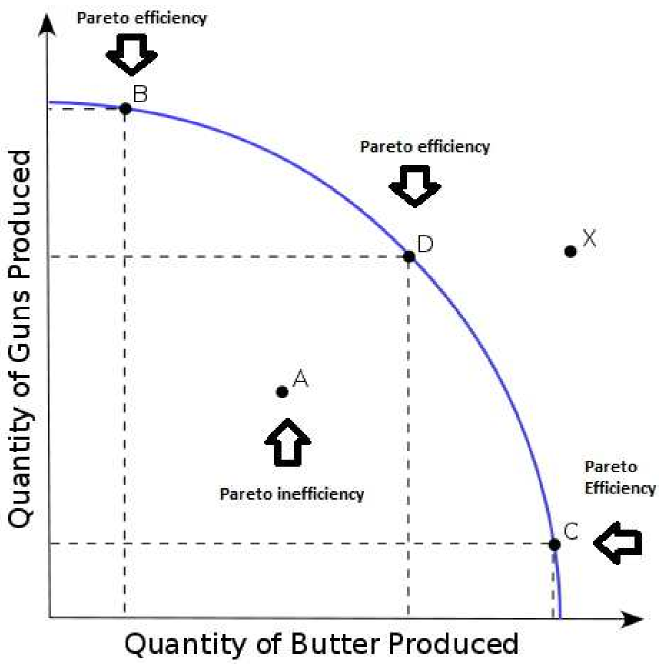
All I need to do to this end is to remind the reader to keep in mind that info-gap's robustness model is a simple instance of Wald's Maximin model. This means of course that because an instance of a prototype cannot possibly generalize the prototype, info-gap decision theory cannot possibly generalize Wald's Maximin paradigm.

The implication of this is that the claim that info-gap's alleged generalization of Wald's Maximin strategy 'permits' the generation of 'robustness curves' is doubly in error hence misleading. Not only is it false on the "generalization" claim, it misrepresents the facts about the capabilities of Wald's Maximin. It thus gives the badly misleading impression that, unlike Wald's Maximin theory, info-gap decision theory does 'permit' the construction of such curves.

The fact of the matter is of course that the same procedure/method that is used to create 'robustness curves' in the framework of an info-gap robustness model can be used to create such curves in the framework of a Maximin model. And this is so simply because this procedure has got nothing to do with the Maximin model as such nor with info-gap's robustness model as such.

That is, the 'robustness curves' that the authors mistakenly attribute to info-gap's capabilities as a "generalized Maximin paradigm" are no more and no less than simple instances of what are known universally as *Pareto Efficiency* curves. The typical *Pareto Efficiency curve* shown in Figure J.1 and the explanatory text are taken from Wikipedia.

Looking at the Production-possibility frontier, shows how productive efficiency is a precondition for Pareto efficiency. Point A is not efficient in production because you can produce more of either one or both goods (Butter and Guns) without producing less of the other. Thus, moving from A to D enables you to make one person better off without making anyone else worse off (rise in Pareto efficiency). Moving to point C from point A, however, is not Pareto efficient, as fewer guns are produced. Likewise, moving to point B from point A is not Pareto efficient, as less butter is produced. A point on the frontier curve with the same x or y coordinate will be Pareto efficient.



Source: http://en.wikipedia.org/wiki/Pareto_efficiency

Figure J.1: A typical Pareto Efficiency Curve

One wonders, therefore, on what grounds do Wintle et al. (2011) state that info-gap generalizes Wald's maximin strategy? And on what ground do they insinuate that Wald's Maximin strategy does not 'permit' the construction of 'robustness curves'?

The reason for this blunder may well be due to the fact that the connection between info-gap's robustness curves and *Pareto Optimization* notably *Pareto Efficiency* is not even noted in the info-gap literature including its primary texts Ben-Haim (2001, 2006, 2010)[4]. As a consequence, **info-gap adherents labor under the misconception** that the so-called "info-gap robustness curves" are unique to info-gap indeed, that they are an info-gap innovation. This is yet another illustration of how cut off the info-gap literature is from areas of expertise that bear directly on what info-gap decision theory claims to be doing. Thus, the discussion in Ben-Haim (2001, 2006, 2010) is not only oblivious to info-gap decision theory's relation to *Pareto Optimization* (Pardalos et al. 2008), it is totally oblivious to its relation to *Robust Optimization* (Ben-Tal et al. 2009) .

I should also point out that Wintle et al. (2011) continue to propound the myth that info-gap decision theory is suitable for the management of *severe uncertainty* of the type that it postulates (emphasis added):

> Because climate adaptation strategies will be developed under **severe uncertainty**, it is critical to incorporate uncertainty in decisions using a method such as info-gap, and plan for reducing uncertainty by learning about management effectiveness and other key parameters.
>
> Wintle et al. (2011, p. 358)

Thus, readers of Wintle et al. (2011) are being doubly mislead by this statement. Not only are they being misinformed about the capabilities of info-gap's robustness model, they are given no clue to work out for themselves why "...a method such as info-gap" is the wrong method for the task. That is, readers would have no clue that "...a method such as info-gap" is in fact utterly unsuitable for this task because its robustness model is inherently a model of *local* robustness. Namely, it defines robustness as the **smallest** perturbation in a given nominal value of the parameter of interest that can cause a violation of the performance constraint. I therefore remind the reader of Hayes' (2011, p. 88) recent interesting observation (emphasis added):

> Analysts who were attracted to IGT because they are very uncertain, and hence reluctant to specify a probability distribution for a model's parameters, may be **disappointed** to find that they need to specify the plausibility of possible parameter values in order to identify a robust management strategy.

Apparently, unlike Hayes (2011), Wintle et al. (2011) are not disappointed at all. Indeed, they have no qualms whatsoever about using a model of local robustness that operates in the neighborhood of a point estimate of the parameter of interest, without specifying the likelihood of possible parameter values.

And recall Rout et al.'s (2009, p 785) reflections on this issue:

[4]Given that *Pareto efficiency* is a central concept in *economics,* it is inexcusable that the connection between info-gap's robustness curves and this fundamental concept is not so much as mentioned in the latest book on info-gap decision theory, namely *Info-Gap Economics: An Operational Introduction,* (Ben-Haim, 2010).

> Thus, the method challenges us to question our belief in the nominal estimate, so that we evaluate whether differences within the horizon of uncertainty are 'plausible'. Our uncertainty should not be so severe that a reasonable nominal estimate cannot be selected.

In contrast, Wintle et al. (2011) have not the slightest concern that under severe uncertainty of the type stipulated by info-gap decision theory the point estimate's poor quality may not justify the use of a model of local robustness.

But . . . this should come as no surprise. Because, as indicated above, this stance is very much of a piece with what is being advocated in Wintle et al. (2010). Indeed, Wintle et al. (2010) even goes a step further to propose the use of a model of local robustness as a suitable means for dealing with *Black Swans* and *Unknown Unknowns*.

As a final note, I want to point out that my discussion (Section 7.3) on the role of *rhetoric* in the info-gap literature, is given a vivid illustration in both Wintle et al. (2011) and Wintle et al. (2010).

Appendix K

The Australian info-gap scene

For the reader's convenience I provide a (rather incomplete) overview of info-gap related activities in Australia, including Australian[1] publications on info-gap decision theory, most of which are peer-reviewed articles.

Articles/books

- Adams, V.M. and Pressey, R.L. (2011) An info-gap model to examine the robustness of cost-efficient budget allocations. *ICVRAM 2011: 1st International Conference on Vulnerability and Risk Assessment and Management,* April 11-13, 2011, University of Maryland, College Park, pp. 971-979.
- Beger, M., Grantham, H.S., Pressey, R.L., Wilson, K.A., Peterson, E.L., Dorfman, D., Mumby, P.J., Lourival, R., Brumbaugh, D.R., and Possingham, H.P. (2010) Conservation planning for connectivity across marine, freshwater, and terrestrial realms. *Biological Conservation,* 143(3), 565-575.
- Beresford-Smith, B., and Thompson, C.J. (2007) Managing credit risk with info-gap uncertainty. *Journal of Risk Finance,* 8(1), 24-34.
- Beresford-Smith, B., and Thompson, C.J. (2009) An info-gap approach to managing portfolios of assets with uncertain returns. *Journal of Risk Finance,* 10(3), 277-287.
- Burgman, M.A. (2005) *Risks and Decisions for Conservation and Environmental Management.* Cambridge University Press, Cambridge.
- Burgman, M.A. (2008) Shakespeare, Wald and decision making under uncertainty. *Decision Point,* 23, 8.
- Burgman, M.A., Lindenmayer, D.B., and Elith, J. (2005) Managing landscapes for conservation under uncertainty. *Ecology,* 86(8), 2007-2017.
- Burgman, M.A., Wintle, B.A., Thompson, C.J., Moilanen, A., Runge, M.C., and Ben-Haim, Y. (2010) Reconciling uncertain costs and benefits in Bayes nets for invasive species management. *Risk Analysis: An International Journal,* 30(2), 277-284.
- Davidovitch, L., Stoklosa, R., Majer, J., Nietrzeba, A., Whittle, P., Mengersen, K., and Ben-Haim, Y. (2009) Info-Gap theory and robust design of surveillance for invasive species: The

[1] At least one of the co-authors is based in Australia

case study of Barrow Island. *Journal of Environmental Management,* 90(8), 2785-2793.

· Fischer, J., Peterson, G.D., Gardner, T.A., Gordon, L.J., Fazey, I., Elmqvist, T., Felton, A., Folke, C., and Dovers, S. (2009) Integrating resilience thinking and optimisation for conservation. *Trends in Ecology and Evolution,* 24(10), 549-554.

· Fox, D.R., Ben-Haim, Y., Hayes, K.R., McCarthy, M., Wintle, B., Dunstan, P. (2007) An info-gap approach to power and sample size calculations. *Environmetrics,* 18, 189-203.

· Fox, D. (2008) To IG or not to IG? – that is the question Decision-making under uncertainty. *Decision Point,* 24, 10-11.

· Franklin, J., Sisson, S.A., Burgman, M.A., and Martin, J.K. (2008) Evaluating extreme risks in invasion ecology: learning from banking compliance. *Diversity and Distributions,* 14, 581-591.

· Halpern, B.S., Regan, H.M., Possingham, H.P., and McCarthy, M.A. (2006) Accounting for uncertainty in marine reserve design. *Ecology Letters,* 9, 2-11.

· Halpern, B.S., Regan, H.M., Possingham, H.P., and McCarthy, M.A. (2006a) Rejoinder: Uncertainty and decision making. *Ecology Letters,* 9. 13-14.

· Hayes, K.R., Regan, H.M., and Burgman, M.A. (2007) Introduction to the Concepts and Methods of Uncertainty Analysis. Chapter 7 in *Environmental Risk Assessment of Genetically Modified Organisms: Vol. 3. Transgenic Fish in Developing Countries* (eds A. R. Kapuscinski et al.), 188-208.

· Hayes, K.R. (2011). *Uncertainty and Uncertainty Analysis Methods.* Final report for the Australian Centre of Excellence for Risk Assessment, CSIRO Division of Mathematics, Informatics and Statistics, Hobart, Australia, 130 pp.

· McCarthy, M.A., Lindenmayer, D.B. (2007) Info-gap decision theory for assessing the management of catchments for timber production and urban water supply. *Environmental Management,* 39 (4), 553-562.

· McDonald-Madden, E., Baxter, P.W.J., and Possingham, H.P. (2008) Making robust decisions for conservation with restricted money and knowledge. *Journal of Applied Ecology,* 45, 1630-1638.

· Moilanen, A., Wintle, B.A., Elith, J., and Burgman, M. (2006) Uncertainty analysis for regional-scale reserve selection. *Conservation Biology,* 20(6), 1688-1697.

· Moilanen, A. and Wintle, B.A. (2006a) Uncertainty analysis favours selection of spatially aggregated reserve structures. *Biological Conservation,* 129(3), 427-434.

· Moilanen, A., Runge, M.C., Elith, J., Tyre, A., Carmel, Y., Fegraus, E., Wintle, B., Burgman, M., and Ben-Haim, Y. (2006b) Planning for robust reserve networks using uncertainty analysis. *Ecological Modelling,* 199(1), 115-124.

· Nicholson, E. and Possingham, H.P. (2007) Conservation planning for the persistence of multiple species under uncertainty: an application of info-gap decision theory. *Ecological Applications,* 17, 251-265

· Regan, H.M., Ben-Haim, Y., Langford, B., Wilson, W.G., Lundberg, P., Andelman, S.J., Burgman, M.A., (2005) Robust decision making under severe uncertainty for conservation management. *Ecological Applications,* 15(4), 1471-1477.

· Rochelle, C. (2008) Assessing and managing environmental risks. *Minerva,* 46, 275-283.

· Rout, T.M., Thompson, C.J., and McCarthy, M.A. (2009) Robust decisions for declaring eradication of invasive species. *Journal of Applied Ecology,* 46, 782-786.

· Sniedovich, M. (2007) The art and science of modeling decision-making under severe uncertainty. *Decision Making in Manufacturing and Services,* 1-2, 111-136.

· Sniedovich, M. (2008) A call for the reassessment of Info-Gap decision theory. *Decision Point,* 24, 10.

· Sniedovich, M. (2008a) From Shakespeare to Wald: modeling wors-case analysis in the face of severe uncertainty. *Decision Point,* 22, 8-9.

· Sniedovich, M. (2008b) Wald's Maximin model: a treasure in disguise! *Journal of Risk Finance,* 9(3), 287-291.

· Sniedovich, M. (2009) Modeling of robustness against severe uncertainty, pp. 33- 42, *Proceedings of the 10th International Symposium on Operational Research,* SOR'09, Nova Gorica, Slovenia, September 23-25, 2009.

· Sniedovich, M. (2009a) A Critique of Info-Gap Robustness Model. In *Safety, Reliability and Risk Analysis: Theory, Methods and Applications,* Martorell et al. (eds), pp. 2071-2079, Taylor and Francis Group, London.

· Sniedovich, M. (2010) A bird's view of Info-Gap decision theory. *Journal of Risk Finance,* 11(3), 268-283.

· Sniedovich, M. (2011) Black swans, new Nostradamuses, voodoo decision theories and the science of decision-making under severe uncertainty. *International Transactions in Operational Research,* in press.

· Sniedovich, M. (2011a) A classic decision theoretic perspective on worst-case analysis. *Applications of Mathematics,* in press.

· Thompson, C.J., Guttmann, A.J., and Thompson, B.J.P. (2008) Trading indicators with information-gap uncertainty. *Journal of Risk Finance,* 9(5), 467-476.

· Walshe, T. and Massenbauer, T. (2008) Decision-making under climatic uncertainty: A case study involving an Australian Ramsar-listed wetland. *Ecological Management and Restoration,* 9(3), 202-208.

· Wintle, B.A., Runge, M.C., and Bekessy, S.A. (2010) Allocating monitoring effort in the face of unknown unknowns. *Ecology Letters,* 13(11), 1325-1337.

· Wintle, B.A. (2010) On the lookout for black swans: monitoring for unknowns unknowns. *Decision Point,* 43, 4-7.

· Wintle, B.A., Bekessy, S.A., Keith, D.A., van Wilgen, B.W., Cabeza, M., Schroder, B., Carvalho, S.B., Falcucci, A., Maiorano, L., Regan, T.J., Rondinini, C., Boitani, L. and Possingham, H.P. (2011) Ecological-economic optimization of biodiversity conservation under climate change. *Nature Climate Change,* Volume 1, 355-359.

· Yokomizo, H., Possingham, H.P., Thomas, M.B., and Buckley, Y. M. (2009) Managing the impact of invasive species: the value of knowing the density-impact curve. *Ecological*

Applications, 19(2), 376-386.

PhD theses:

- Emily Nicholson (2006) Planning for persistence: using the risk of extinction in multi-species approaches to conservation planning. The Ecology Centre, University of Queensland.
- Tracy Rout (2009) Monitoring and managing species when their presence is uncertain. Department of Botany, University of Melbourne.

Honours theses:

- Jaeger, Renn-Jones. (2007) Robust Decision-Making Under Uncertainty. Department of Mathematics and Statistics, The University of Melbourne.
- Daphne, Do. (2008) Formulating and Modelling Robust Decision-Making Problems Under Severe Uncertainty. Department of Mathematics and Statistics, The University of Melbourne.

ACERA Endorsed project reports with some info-gap content:[2]

- 0601_0611: Reconciling uncertain costs and benefits in Bayes nets for invasive species management.
- 0602: Assessment of strategies for evaluating extreme risks.
- 0604: Optimal allocation of resources to emergency response actions for invasive species.
- 0605: Statistical methods for biosecurity monitoring and surveillance.
- 0705A: Uncertainty and uncertainty analysis methods.
- 0706: Evaluating vegetation condition measures for cost effective biodiversity investment planning.

Workshops:

- Breakfast workshop: Managing Financial Risks with Uncertainty. Investment Centre Victoria, Melbourne, (organized by MASCOS, AMSI), Melbourne, November 22, 2005.
- Breakfast workshop: Managing Financial Risks with Uncertainty. New South Wales Trade and Investment Centre, (organized by MASCOS, AMSI), Sydney, July 25, 2006.
- 1-Day Workshop on Info-Gap Theory and Its Applications in Biological Conservation, University of Melbourne, Melbourne, August 2, 2007.
- 5-Day Workshop on Info-Gap Theory and Its Applications in Biological Conservation, University of Queensland, Brisbane, September, 15-19, 2008.

[2] Can be downloaded at http://www.acera.unimelb.edu.au/materials/core.html.

Appendix L

Errata

I plan to maintain an errata page for this document on my website. For details, see

http://info-gap.moshe-online.com/acera.html

Bibliography

[1] Adams, V.M., and Robert L. Pressey, R.L. (2011) An info-gap model to examine the robustness of cost-efficient budget allocations. ICVRAM 2011: 1st International Conference on Vulnerability and Risk Assessment and Management, April 11-13, 2011, University of Maryland, College Park, pp. 971-979.

[2] Aghassi, M. and Bertsimas, D. (2006) Robust game theory. *Mathematical Programming,* 107(1), 231-273.

[3] Ahn, D.S. (2008) Ambiguity without a state space. *Review of Economic Studies,* 75, 3-28.

[4] Akram, Q.F., Ben-Haim, Y., and Eitrheim, Ø. (2006) Managing uncertainty through robust-satisficing monetary policy. Norges Bank Working paper ANO 2006/10.

[5] Bai, D., Carpenter, T., and Mulvey, J. (1997) Making a case for robust optimization models. *Management Science,* 43, 895-907.

[6] Beigi, G., Hosseinzadeh, L.F., Noora, A.A., Mozaffari, M.R., Gholami, K., and Deghghan, F. (2009) Sensitivity and stability analysis in DEA on interval data by using MOLP methods. *Applied Mathematical Sciences,* 3(8), 891-908.

[7] Ben-Haim, Y. (1994) Convex models of uncertainty: applications and implications. *Erkenntnis,* 4, 139-156.

[8] Ben-Haim, Y. (1995) A non-probabilistic measure of reliability of linear systems based on expansion of convex models. *Structural Safety,* 17, 91-109.

[9] Ben-Haim, Y. (1996) *Robust Reliability in the Mechanical Sciences.* Springer, Berlin.

[10] Ben-Haim, Y. (1999) Design certification with information-gap uncertainty, *Structural Safety,* 2, 269-289.

[11] Ben-Haim, Y. (2001) *Information Gap Decision Theory.* Academic Press, Amsterdam.

[12] Ben-Haim, Y. (2006) *Info-Gap Decision Theory.* Elsevier, Amsterdam.

[13] Ben-Haim, Y. (2007) Frequently asked questions about info-gap decision theory. http://www.technion.ac.il/yakov/IGT/faqs01.pdf

[14] Ben-Haim, Y. (2007a) Info-gap robust satisficing and the probability of survival. De Nederlandsche Bank (DNB) Working Paper No. 138/2007.

[15] Ben-Haim, y. (2009) Info-gap forecasting and the advantage of suboptimal models. *European Journal of Operational Research,* 197(1), 203-213.

[16] Ben-Haim, Y. (2010) *Info-gap Economics: An Operational Introduction.* Palgrave.

[17] Ben-Haim, Y. (2011) Interpreting null results from measurements with uncertain correlations: an info-gap approach. *Risk Analysis,* 31(1), 78-85.

[18] Ben-Haim, Y., and Cogan, S. (2011). Linear Bounds on an Uncertain Non-Linear Oscillator: An Info-Gap Approach. *IUTAM Bookseries,* volume 27, pp. 3-14.

[19] Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2006) *Mathematical Programming*, Special issue on *Robust Optimization,* 107(1-2).

[20] Ben-Tal, A., Boyd, S., Nemirovski, A. (2006a) Extending scope of robust optimization: comprehensive robust counterparts of uncertain problems. *Mathematical Programming* Series B, 107, 63-89.

[21] Ben-Tal, A., El Gaoui, L., and Nemirovski, A. (2009) *Robust Optimization.* Princeton University Press, Princeton.

[22] Ben-Tal, A., Golany, B., Shtern, S. (2009b) Robust multi-echelon multi-period inventory control. *European Journal of Operational Research* 199(3), 922-935.

[23] Ben-Tal, A., Davidson, W., Bertsimas, D., and Brown, D.B. (2010). A soft robust model for 9 optimization under ambiguity. *Operations Research,* 58(4):1220-1234.

[24] Bertsimas, D., and Sim, M. (2004) The price of robustness. *Operations Research,* 52(1), 35-53.

[25] Bertsimas, D., and D.B. Brown. D.B. (2009). Constructing uncertainty sets for robust linear optimization. *Operations Research,* 57(6):1483-1495.

[26] Bramley, M., Gouldby, B., Hurford, A., Flikweert, J-J., Roca Collell, M., Sayers, P., Simm, J., and Wallis, M. (2009) *PAMS (Performance-based Asset Management System) – Phase 2 Outcome Summary Report.* Report SC040018/R1, Product code: SCHO1209BRRA-E-E, Environment Agency, UK.
http://publications.environment-agency.gov.uk/epages/eapublications.storefront/4d72dd4801b700e6273fc0a80296066c/Product/View/SCHO1209BRRA-E-E#

[27] Brock, W.A., and Durlauf, S.N. (2005) Local robustness analysis: theory and application. *Journal of Economic Dynamics & Control,* 29, 2067-2092.

[28] Broens, D-F., and Haneveld, W.K.K. (1995) Investment evaluation based on the commercial scope: the production of natural gas. *Annals of Operations Research,* 59(1), 195-226.

[29] Buhl-Mortensen ,L. (1996) Type-II Statistical Errors in Environmental Science and the Precautionary Principle. *Marine Pollution Bulletin,* 32(7), 528-531.

[30] Burgman, M.A. (2005) *Risks and Decisions for Conservation and Environmental Management.* Cambridge University Press, Cambridge.

[31] Burgman, M.A. (2008) Shakespeare, Wald and decision making under uncertainty. *Decision Point,* 23, 8.

[32] Burgman, M.A., Wintle, B.A., Thompson, C.J., Moilanen, A., Runge, M.C., and Ben-Haim, Y. (2008) Reconciling uncertain costs and benefits in Bayes nets for invasive species management. ACERA Endorsed Core Material: Final Report, Project 0601 – 0611, July 2008.
http://www.acera.unimelb.edu.au/materials/endorsed/0601_0611.pdf (Downloaded on March 11, 2011)

[33] Burgman, M.A., Wintle, B.A., Thompson, C.J., Moilanen, A., Runge, M.C., and Ben-Haim, Y. (2010) Reconciling uncertain costs and benefits in Bayes nets for invasive species management. *Risk Analysis: An International Journal,* 30(2), 277-284.

[34] Carrasco, L. R., Baker, R., MacLeod, A., Knight, J.D., and Mumford, J.D. (2010) Optimal and robust control of invasive alien species spreading in homogeneous landscapes. *Journal of the Royal Society Interface,* 7(44), 529-540.

[35] Cohen, M., and Jaffray, J-Y. (1980) Rational behavior under complete ignorance. *Econometrica,* 48, 1281-1299.

[36] Costanza, R. (1989). What is ecological economics? *Ecological Economics,* 1(1):1-7.

[37] Cull, P. (1981) Global stability of population models. *Bulletin of Mathematical Biology,* 43, 47-58.

[38] Davidovitch, L. (2009) *Strategic Interactions under Severe Uncertainty.* PhD Thesis, Department of Mechanical Engineering, Technion, Haifa, Israel.
http://www.graduate.technion.ac.il/Theses/Abstracts.asp?Id=24425

[39] Davidovitch, L., Stoklosa, R., Majer, J., Nietrzeba, A., Whittle, W., Mengersen, K., and Ben-Haim, Y. (2009) Info-gap theory and robust design of surveillance for invasive species: the case study of Barrow Island. *Journal of Environmental Management,* 90(8), 2785-2793.

[40] Davidovitch, L., and Ben-Haim, Y. (2011) Is your profiling strategy robust? *Law, Probability and Risk,* in press 2011.

[41] D.G. de Figueiredo, D.G., Miyagaki, O.H., and Ruf, B. (1995) Elliptic equations in $\mathbf{R}^2$ with nonlinearities in the critical growth range. *Calculus of Variations and Partial Differential Equations,* 3(2), 139-153.

[42] Demyanov, V.M., and Malozemov, V.N. (1990) *Introduction to Minimax.* Dover, NY.

[43] A. Deines, A., Peterson, E., Boeckner, D., Boyle, J., Keighley, A., Kogut, J., Lubben, J., Rebarber, R., Ryan, R., Tenhumberg, B., Townley, S., Tyre, A.J. (2007) Robust population management under uncertainty for structured population models. *Ecological Applications,* 178, 2175-2183.

[44] Do, D. (2008) Formulating and Modelling Robust Decision-Making Problems Under Severe Uncertainty. Honours Thesis, Department of Mathematics and Statistics, The University of Melbourne.

[45] Du, D.Z., and Pardalos, P.M. (1995) *Minimax and Applications.* Springer Verlag, Berlin.

[46] Eiselt, H.A. and Langley A. (1990) Some extensions of domain criteria in decision making under uncertainty. *Decision Sciences,* 21, 138-153.

[47] Eiselt, H.A., Sandblom, C.L., and Jain, N. (1998) A Spatial Criterion as Decision Aid for Capital Projects: Locating a Sewage Treatment Plant in Halifax, Nova Scotia. *Journal of the Operational Research Society,* 49(1), 23-27.

[48] Emelichev V.A., and Karelkina O.V. (2009) Finite cooperative games: parametrisation of the concept of equilibrium (from Pareto to Nash) and stability of the efficient situation in the Hölder metric. *Discrete Mathematics and Applications,* 19(3), 229-236.

[49] Fox, D.R., Ben-Haim, Y., Hayes, K.R., McCarthy, M., Wintle, B., Dunstan, P. (2007) An info-gap approach to power and sample size calculations. *Environmetrics,* 18, 189-203.

[50] Franklin, J., Sisson, S.A., Burgman, M.A., and Martin, J.K. (2008) Evaluating extreme risks in invasion ecology: learning from banking compliance. *Diversity and Distributions,* 14, 581-591.

[51] French, S.D. (1988) *Decision Theory.* Ellis Horwood, NY.

[52] Ghirardato, P. and Marinacci, M. (2002) Ambiguity made precise: a comparative foundation. *Journal of Economic Theory,* 102, 251-289.

[53] Graillat, S. and Langlois, P. (2004) Computation of stability radius for polynomials. Preprint No31, Laboratoire MANO, The University of Perpignan, France. http://webdali.univ-perp.fr/publis/Icons/pdf.gif

[54] Gilboa, I. (2009) *Theory of Decision Under Uncertainty.* Cambridge University Press.

[55] Gilboa, I. (2010) *Rational Choice.* The MIT Press.

[56] Giordani, P., and Söderlind, P. (2004) Solution with macromodels with Hansen-Sargent robust policies: some extensions. *Journal of Economic Dynamics and Control,* 28, 2367-2397.

[57] Greenberg, H.J. (1998) An annotated bibliography for post-solution analysis in mixed integer programming and combinatorial optimization. In D.L. Woodruff, D.L. (editor), *Advances in Computational and Stochastic Optimization, Logic Programming, and Heuristic Search,* pp. 97-148. Kluwer.

[58] Hall, J.W., and Ben-Haim, Y. (2007) Making Responsible Decisions (When it Seems that You Can't). Paper posted on the FloodRiskNet.org.uk website. See http://www.floodrisknet.org.uk/a/2007/11/hall-benhaim.pdf/view. Also available at http://www.technion.ac.il/yakov/IGT/aaahall-ybh2007.pdf

[59] Hall, J., and Harvey, H. (2009) Decision making under severe uncertainty for flood risk management: a case study of info-gap robustness analysis. Eighth International Conference on Hydroinformatics January 12-16, 2009, Concepcion, Chile.

[60] Halpern, B.S., Regan, H.M., Possingham, H.P., and McCarthy, M.A. (2006) Accounting for uncertainty in marine reserve design. *Ecology Letters,* 9, 2-11.

[61] Hanley, N. (1992) Are there environmental limits to cost benefit analysis? *Environmental and Resource Economics,* 2(1),33-59.

[62] Hansen, L.P., and Sargent, T.J. (2008) *Robustness.* Princeton University Press.

[63] Hansen, L.P., and Sargent, T.J. (2010) Wanting robustness in macroeconomics. Working paper. http://homepages.nyu.edu/∼ts43/research/wanting.pdf

[64] Harsanyi, J.C. (1976) Can the maximin principle serve as a basis for morality?: A critique of John Rawls's theory. Chapter IV (pp. 37-63) in *Essays on Ethics, Social Behavior, and Scientific Explanation,* Springer Verlag, Berlin.

[65] Hayes, K.R. (2010). *Uncertainty and Uncertainty Analysis Methods.* Final report for the Australian Centre of Excellence for Risk Assessment, CSIRO Division of Mathematics, Informatics and Statistics, Hobart, Australia, 130 pp.

[66] Hemez, F.M., Ben-Haim, Y., and Cogan, S. (2002) Information-gap robustness for the test-analysis and optimization of a nonlinear transient simulation. *Proceedings of 9th AIAA/ISSMO Symposium on Multidisciplinary Analysis and Optimization,* 4-6 September 2002, Atlanta, Georgia.

[67] Hemez, F.M., Doebling, S.W., and Ben-Haim, Y. (2003) Applying information-gap reasoning to the predictive accuracy assessment of transient dynamics simulations. *Proceedings of the 8th International Conference on Recent Advances in Structural Dynamics,* July 14-16, 2003, Southampton, United Kingdom.

[68] Hemez, F.M., and Ben-Haim, Y. (2004) Info-gap robustness for the correlation of tests and simulations of a non-linear transient. *Mechanical Systems and Signal Processing,* 18, 1443-1467.

[69] Herdrix, E.M.T., Mecking, C.J., Hendriks, T.H.B. (1996) Theory and Methodology Finding robust solutions for product design problems. *European Journal of Operational Research,* 92(1), 28-36.

[70] Hildebrandt, P., and Knoke, T. (2011) Investment decisions under uncertainty — A methodological review on forest science studies. *Forest Policy and Economics,* 13(1), 1-15.

[71] Hillier, F.S., and and Lieberman, G.J. (2005) *Introduction to Operations Research.* McGraw-Hill.

[72] Hine, D., and Hall, J.W. (2010) Information gap analysis of flood model uncertainties and regional frequency analysis. *Water Resources Research,* 46, in press.

[73] Hinrichsen, D., and Pritchard, A.J. (1986a) Stability radii of linear systems. *Systems & Conlrol Letters,* 7,1-10.

[74] Hinrichsen, D., and Pritchard, A.J. (1986b) Stability radius for structured perturbations and the algebraic Riccati equation. *Systems & Conlrol Letters,* 8, 105-113.

[75] Hinrichsen, D., and Pritchard, A.J. (1990) A note on some differences between real and complex stability radii. *Systems & Control Letters,*14, 401-409.

[76] Hinrichsen, D., and Pritchard, A.J. (1992) Robustness measures for linear systems with applications to stability radii of Hurwitz and Schur polynomials. *International Journal of Control,* 55:809-844.

[77] Hinrichsen, D., and Pritchard, A.J. (2010) *Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness.* Springer (hard cover, 2005).

[78] Houck, O. (2010) Worst case and the DEEPWATER HORIZON Blowout: there ought to be a law. *Tulane Environmental Law Journal,* 24(1):1-18.

[79] Huber, P.J. (1981) *Robust Statistics.* Wiley.

[80] Huber, P.J., and Ronchetti, E. (2009) *Robust Statistics,* 2nd edition, Wiley.

[81] Hurwicz, L. (1951) *A class of criteria for decision-making under ignorance.* Cowles Commission Discussion Paper: Statistics No. 356.

[82] Kadane, J.B. (2011) *Principles of Uncertainty.* Chapman and Hall.

[83] Keynes, J.M. (1937) The General theory of employment. *Quarterly Journal of Economics,* 51(2), 209-223.

[84] Knight, F.H. (1921) *Risk, Uncertainty, and Profit.* Houghton Mifflin Co.(Re-issued by University of Chicago Press, 1971).

[85] Knoke, T. (2010). Book Reviews: Info-Gap Economics: An Operational Introduction. *Ecological Economics,* 70(3), 567-568.

[86] Kouvelis, P., and Yu, G. (1997) *Robust Discrete Optimization and Its Applications.* Kluwer, Boston.

[87] Krippendorff, K. (1986) *A Dictionary of Cybernetics*. The American Society for Cybernetics, Norfolk VA.

[88] Laskey, K.B., and Lehner, P.E. (1994) Metareasoning and the problem of small worlds. *IEEE Transactions on Systems, Man and Cybernetics,* 24(11), 1643-1652.

[89] Lempert, R.J., Groves, D.G., Popper, S.W., and Bankes, S.C. (2003) *Shaping the Next One Hundred Years: New Methods for Quantitative, Long-Term Policy Analysis.* RAND, Santa Monica, CA.

[90] Lempert, R.J., Groves, D.G., Popper, S.W., and Bankes, S.C. (2006) A General, analytic method for generating robust strategies and narrative scenarios. Management Science, 52(4) 514-528.

[91] Lempert, R.J., and Collins, M.T. (2007) Managing the risk of uncertain threshold responses: comparison of robust, optimum, and precautionary approaches. Risk Analysis, 27(4), 1009-1026.

[92] Lu, S-H., and Fong, I-K. (1998) Stability radius of linear normal distributed parameter systems with multiple directional perturbation. Proceedings of the 37th IEEE Conference on Decision &, Control, Tampa, Florida USA December 1998.

[93] Luce, R.D., and Raiffa, H. (1957) *Games and Decisions: Introduction and Critical Survey*. Wiley.

[94] Milne, W.E., and Reynolds, R.R. (1962) Fifth-order methods for the numerical solution of ordinary differential equations. *Journal of the ACM,* 9(1), 64-70.

[95] Moffitt, L.J., Stranlund, J.K., and Field, B.C. (2005) Inspections to avert terrorism: robustness under severe uncertainty. *Journal of Homeland Security and Emergency Management,* 2(3), 1-17.

[96] Moffitt, L.J., Stranlund, J.K., and Osteen, C.D. (2009) Robust detection protocols for uncertain introductions of invasive species. *Working Paper No. 2009-6.* University of Massachusetts Amherst, Department of Resource Economics.
http://courses.umass.edu/resec/workingpapers/documents/ResEcWorkingPaper2009-6.pdf

[97] Moffitt, L.J., Stranlund, J.K., and Osteen, C.D. (2010) Securing the Border from Invasives: Robust Inspections under Severe Uncertainty.*Economics Research International,* Volume 2010.
2010 doi:10.1155/2010/510127

[98] oilanen, A., Runge, M.C., Elith, J., Tyre, A., Carmel, Y., Fegraus, E., Wintle, B., Burgman, M., and Ben-Haim, Y. (2006) Planning for robust reserve networks using uncertainty analysis. *Ecological Modelling,* 199(1), 115-124.

[99] Mulvey, J,M., Vanderbei, R.J., and Zenios, S.A. (1995) Robust Optimization of Large-Scale Systems. *Operations Research,* 43(2), 264-281.

[100] Nicholson, E. and Possingham, H.P. (2007) Conservation planning for the persistence of multiple species under uncertainty: an application of info-gap decision theory. *Ecological Applications,* 17, 251-265

[101] Packard, A., and Doyle, J.C. (1993) The complex structured singular value. Automatica, 29:71-109.

[102] Paice, A.D.B., and Wirth, F.R. (1998) Analysis of the local robustness of stability for flows. *Mathematics of Control Signals and Systems,* 11, 289-302.

[103] Pardalos, P.M., Migdalas, A., Pitsoulis, L. (Eds.) (2008) *Pareto Optimality, Game Theory and Equilibria.* Springer Optimization and Its Applications, Vol. 17.

[104] Petric, J., and Zlobec S. (1983) Nonlinear Programming. *Naucna,* Knjiga, Belgrade (in Serbo-Croatian).

[105] Possingham, H. (2010) Valē AEDA What's left to say? *Decision Point,* 45, p. 2.

[106] Pugh, C. and Shub, M. (2004) Stable ergodicity. *Bulletin of the American Mathematical Society,* 41, 1-41.

[107] Puppe, C., and Schlag, K.H. (2009). Choice under complete uncertainty when outcome spaces are state-dependent. *Theory and Decision,* 66, 1-16.

[108] Raab, R.L., and Feroz E.H. (2007) A productivity growth accounting approach to the ranking of developing and developed nations. *The International Journal of Accounting,* 42(4), 396-415.

[109] Rawls, J. (2005) *Theory of Justice,* Belknap Press, Cambridge, MA.

[110] Reemstem, R., and Rückmann, J. (1998) *Semi-Infinite Programming.* Kluwer, Boston.

[111] Renn-Jones, J. (2007) Robust Decision-Making Under Uncertainty. Honours Thesis, Department of Mathematics and Statistics, The University of Melbourne.

[112] Resnik, M.D. (1987) *Choices: an Introduction to Decision Theory.* University of Minnesota Press, Minneapolis, MN.

[113] Rodgers, W.H., and Moritz, A.T. (2010). The worst case and the worst example: an agenda for any young lawyer who wants to save the world from climate chaos. *Southeastern Environmental Law Journal,* 17(2):295-326.

[114] Rosenhead, M.J, Elton, M., and Gupta, S.K. (1972) Robustness and optimality as criteria for strategic decisions. *Operational Research Quarterly* 23(4), 413-430.

[115] Rout, T.M., Thompson, C.J., and McCarthy, M.A. (2009) Robust decisions for declaring eradication of invasive species. *Journal of Applied Ecology,* 46, 782-786.

[116] Rustem, B., and Howe, M. (2002) *Algorithms for Worst-case Design and Applications to Risk Management.* Princeton University Press, Princeton.

[117] Savage, L.J. (1951) The theory of statistical decision. *Journal of the American Statistical Associatiion*, 46(253), 55-67.

[118] Savage, L.J. (1954) The Foundation of Statistics. Wiley, NY.

[119] Sayers, P.B., Wallis, M.J., Simm, J.D., Baxter, G., and Andryszewski, T. (2011) Towards the Next Generation of Risk-Based Asset Management Tools. Chapter 15, in Pender, G., and Faulkner, H. (editors). *Flood Risk Science and Management,* Wiley.

[120] Schneller, G.O., and Sphicas, G.P. (1983) Decision making under uncertainty: Starr's domain Criterion, *Theory and Decisionh,* 15, 321-336.

[121] Schwartz, B., Ben-Haim, Y., and Dacso, C. (2011) What Makes a good decision? Robust satisficing as a normative standard of rational decision making. *The Journal for the Theory of Social Behaviour,* in press.

[122] Sion, M. (1958) On general minimax theorems. *Pacific Journal of Mathematics,* 8(1), 171-176.

[123] Sniedovich, M. (2003) OR/MS Games: 3. counterfeit coin problem, *INFORMS Transactions on Education,* 3(2): 1-10.

[124] Sniedovich, M. (2006). What's Wrong with Info-Gap? An Operations Research Perspective. Working Paper No. MS-1-06. Department of Mathematics and Statistics, The University of Melbourne.

[125] Sniedovich, M. (2007) The art and science of modeling decision-making under severe uncertainty. *Decision Making in Manufacturing and Services,* 1(1-2), 111-136.

[126] Sniedovich, M. (2008) Wald's maximin model: a treasure in disguise! Ĵournal of Risk Finance, 9(3), 287-291.

[127] Sniedovich, M. (2008a) FAQs about info-gap decision theory. Working Paper No MS-12-08, http://info-gap.moshe-online.com/faqs_about_infogap.pdf.

[128] Sniedovich, M. (2008b) Wald's Maximin model: a treasure in disguise! *Journal of Risk Finance,* 9(3), 287-291.

[129] Sniedovich, M. (2008c) A call for the reassessment of Info-Gap decision theory. *Decision Point,* 24, p. 10.

[130] Sniedovich, M. (2009) Modeling of robustness against severe uncertainty, pp. 33- 42, *Proceedings of the 10th International Symposium on Operational Research,* SOR'09, Nova Gorica, Slovenia, September 23-25, 2009.

[131] Sniedovich, M. (2009a) A Critique of Info-Gap Robustness Model. In *Safety, Reliability and Risk Analysis: Theory, Methods and Applications,* Martorell et al. (eds), pp. 2071-2079, Taylor and Francis Group, London.

[132] Sniedovich, M. (2010) A bird's view of info-gap decision theory. *Journal of Risk Finance,* 11(3), 268-283.

[133] Sniedovich, M. (2011) Black swans, new Nostradamuses, voodoo decision theories and the science of decision-making under severe uncertainty. *International Transactions in Operational Research,* in press.

[134] Sniedovich, M. (2011a) A classic decision theoretic perspective on worst-case analysis. *Applications of Mathematics,* in press.

[135] Sotskov, Y.N. (2007) Recent results on stability analysis of an optimal assembly line balance. *International Journal "Information Theories & Applications,* 14,174-181.

[136] Sotskov, Y.N., Leontev, I.V.K., Gordeev, E.N. (1995). Some concepts of stability analysis in combinatorial optimization. *Discrete Applied Mathematics,* 58, 169-190.

[137] Sotskov, Y.N., Sotskova, N.Y., and Werner, F. (1997) Stability of an optimal schedule in a job shop. *Omega,* 25(4), 397-414.

[138] Sotskov, Y.N., Tanaev, V.S., and Werner, F. (1998) Stability radius of an optimal schedule: a survey and recent developments. In Yu, G. (editor) *Industrial Applications of Combinatorial Optimization,* pp. 72-108. Kluwer.

[139] Sotskov, Y.N., Wagelmans, A.P.M., and Werner, F. (1998) On the calculation of the stability radius of an optimal or an approximate schedule. *Annals of Operations Research,* 83, 213-252.

[140] Starr, M.K. (1963) *Product design and decision theory.* Prentice-Hall, Englewood Cliffs, NJ.

[141] Starr, M. K. (1966) A Discussion of some normative criteria for decision-making under uncertainty. *Industrial Management Review,* 8(1), 71-78.

[142] Stigler, S.M. (2010) The changing history of robustness. *The American Statistician,* 64(4), 277-281.

[143] Stoye, J. (2009) Minimax. In S. N. Durlauf, and L. E. Blume (Eds.), *The New Palgrave dictionary of economics.* London: Palgrave Macmillan.

[144] Stoye, J. (2009a) Minimax regret. In S. N. Durlauf, and L. E. Blume (Eds.), *The New Palgrave dictionary of economics.* London: Palgrave Macmillan.

[145] Stoye, J. (2011) Statistical decisions under ambiguity. *Theory and Decision,* 70(2), 129-148.

[146] Taleb, N.N. (2007) *The Black Swan: The Impact of the Highly Improbable.* Random House.

[147] Taleb, N.N. (2005) *Fooled by Randomness.* Random House.

[148] Taleb, N.N., and Blyth, M. (2011) The Black Swan of Cairo. *Foreign Affairs,* 90(3), 33-39.

[149] Tintner, G. (1952) Abraham Wald's contributions to econometrics. *The Annals of Mathematical Statistics,* 23(1), 21-28.

[150] Tolstobrov, S.S. (1998) On the determination of the stability radius of a polynomial. *Journal of Automation and Information Sciences,* 29(2), 51-64.

[151] Ullman, D.G. (2006) *Making Robust Decisions: Decision Management For Technical, Business, & Service Teams.* Trafford Publishing.

[152] Vardas, G., and Xepapadeas, A. (2010) Model uncertainty, ambiguity and the precautionary principle: implications for biodiversity management. *Environmental Resources and Resource Economics,* 45, 379-404.

[153] Vladimirou, H., and Zenios, S.A. (1997) Stochastic Programming and Robust Optimization. Chapter 12 in *Advances in Sensitivity Analysis and Parametric Programming*, edited by Gal, T., and Greenberg, H.J, Kluwer, Boston.

[154] von Neumann, J. (1928) Zur theories der gesellschaftsspiele, *Math. Annalen*, Volume 100, 295-320.

[155] von Neumann, J., and Morgenstern, O. (1944) *Theory of Games and Economic Behavior.* Princeton University Press, Princeton.

[156] Wald, A. (1939) Contributions to the theory of statistical estimation and testing hypotheses. *Annals of Mathematical Statistics* 10(4), 299-326.

[157] Wald, A. (1945) Statistical decision functions which minimize the maximum risk, *The Annals of Mathematics,* 46(2), 265-280.

[158] Wald, A. (1950) *Statistical Decision Functions.* John Wiley, NY.

[159] Wilf, H.S. (1960) Maximally stable numerical integration. *Journal of the Society for Industrial and Applied Mathematics* 8(3), 537-540.

[160] Williams, N. (2008) Entry *Robust Control* in the *New Palgrave Dictionary of Economics,* Second Edition.
http://www.dictionaryofeconomics.com/article?id=pde2008_R000250&q=//robust %20control&topicid=&result_number=1

[161] Winston, W.L. (2004) *Operations Research: Applications and Algorithms.* Thompson.

[162] Wintle, B.A., Runge, M.C., and Bekessy, S.A. (2010) Allocating monitoring effort in the face of unknown unknowns. *Ecology Letters,* 13(11), 1325-1337.

[163] Wintle, B.A. (2010) On the lookout for black swans: monitoring for unknowns unknowns. *Decision Point,* 43, 4-7.

[164] Wintle, B.A., Bekessy, S.A., Keith, D.A., van Wilgen, B.W., Cabeza, M., Schroder, B., Carvalho, S.B., Falcucci, A., Maiorano, L., Regan, T.J., Rondinini, C., Boitani, L. and Possingham, H.P. (2011) Ecological-economic optimization of biodiversity conservation under climate change. *Nature Climate Change,* Volume 1, 355-359.

[165] Wong, H-Y., and Rosenhead, J. (2000) A rigorous definition of robustness analysis. *Journal of the Operational Research Society,* 51(2), 176-182.

[166] Yemshanov, D., Koch, F.H., Ben-Haim, Y., and Smith, W.D. (2010) Robustness of Risk Maps and Survey Networks to Knowledge Gaps About a New Invasive Pest. *Risk Analysis,* 30(2), 261-276.

[167] Yokomizo, H., Possingham, H.P., Thomas, M.B., and Buckley, Y. M. (2009) Managing the impact of invasive species: the value of knowing the density-impact curve. *Ecological Applications,* 19(2), 376-386.

[168] Yu, G. (1998) *Industrial Applications of Combinatorial Optimization.* Kluwer.

[169] Zhang, Y. (2007) General robust-optimization formulation for nonlinear programming. *Journal of Optimization Theory and Applications,* 132(1), 111-124.

[170] Zhao, K-Y., and Stoustrup, J. (1997) Computation of the maximal robust H performance radius for uncertain discrete time systems with nonlinear parametric uncertainties. *International Journal of Control,* 61(1), 33-44.

[171] Zhukosky, V.I., and Salukvadze, M.E. (1994) *The Vector-Valued Maximin.* Academic Press, NY.

[172] Zlobec, S. (1987) Survey of input optimization. *Optimization,* 18, 309-348.

[173] Zlobec, S. (1988) Characterizing optimality in mathematical programming models. *Acta Applicandae Mathematicae,* 12, 113-180.

[174] Zlobec, S. (2001) Nondifferentiable optimization: Parametric programming. In *Encyclopedia of Optimization*, Floudas C.A., and Pardalos, P.M. editors, Springer.

[175] Zlobec, S. (2009) Nondifferentiable optimization: Parametric programming. In *Encyclopedia of Optimization*, pp. 2607-2615, Floudas, C.A., and Pardalos, P.M. editors, Springer.

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