The Fundamental Flaws in Info-Gap Decision Theory

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Abstract

- How do you make responsible (robust?!) decisions in the face of (severe?!) uncertainty?
  - Info-Gap decision theory
  - Classical decision theory
  - Robust decision-making
  - Voodoo decision theory
- Australian perspective

This is a

Maths Classification G

presentation.

Maths Classification MA +18

versions can be found at

moshe-online.com

Programme

1. Introduction
2. Info-Gap Decision Theory
3. Classical Decision Theory
4. Robust Decision Making
5. Voodoo Decision Theory
6. Info-Gap Revisited
7. Conclusions
8. Off the record

Motivation

What is the most popular methodology for robust decision-making under severe uncertainty in a number of prestigious organizations in

Australia

?

Motivation

Planning for robust reserve networks using uncertainty analysis

. . . In summary, we recommend info-gap uncertainty analysis as a standard practice in computational reserve planning. The need for robust reserve plans may change the way biological data are interpreted. It also may change the way reserve selection results are evaluated, interpreted and communicated. Information-gap decision theory provides a standardized methodological framework in which implementing reserve selection uncertainty analyses is relatively straightforward. We believe that alternative planning methods that consider robustness to model and data error should be preferred whenever models are based on uncertain data, which is probably the case with nearly all data sets used in reserve planning . . .

Ecological Modelling, I99, pp. 115-124, 2006

Motivation

New Secret Weapon Against Severe Uncertainty

$$\hat{\alpha}(q) := \max\{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha,\tilde{u})\}, q \in \mathcal{Q}$$

Also known as

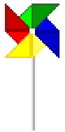
Info-Gap Robustness Model

Very popular in a number of prestigious organizations in Australia

Motivation

Objective of this seminar

- Describe Info-Gap
- Explain why it is fundamentally flawed
- Report on the state of the art
- Report on Moshe's Info-Gap Campaign
- Raise/Answer questions



The Spin stops here!

Info-Gap

Robustness Model

$$\hat{\alpha}(q,r) := \max\{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha,\tilde{u})\}, q \in \mathcal{Q}$$

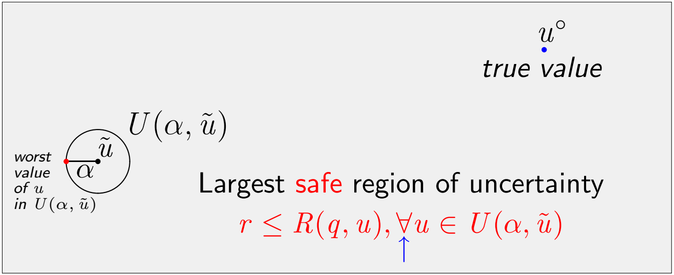
- Decision Space: $\mathcal{Q}$
- Parameter: $u \in \mathcal{U}$, (true value s.t. severe uncertainty)
- Estimate: $\tilde{u} \in \mathcal{U}$
- Uncertainty regions: $U(\alpha,\tilde{u}) \subseteq \mathcal{U}, \alpha \geq 0$
- Performance function R
- Critical performance level r

Info-Gap

Complete Robustness Model

$$\hat{\alpha}(r) := \max_{q\in\mathcal{Q}} \max\{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha,\tilde{u})\}$$

Region of Severe Uncertainty, $\mathcal{U}$

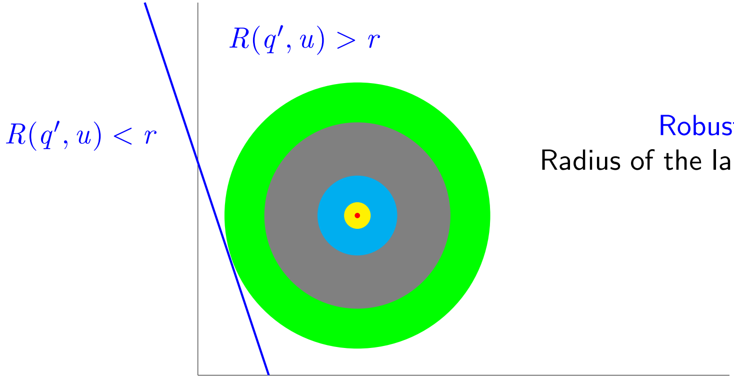


Info-Gap: example

$$\hat{\alpha}(r) = \max_{q \in \mathcal{Q}} \max \{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha, \tilde{u})\}$$

$$U(\alpha, \tilde{u}) = \{u \in \mathbb{R}^2 : (u_1 - \tilde{u}_1)^2 + (u_2 - \tilde{u}_2)^2 \leq \alpha^2\}$$

$$R(q', u) = 3u_1 + u_2 \quad ; \quad r = 4 \quad ; \quad \tilde{u} = (3,3)$$



Robustness:
Radius of the largest safe circle

Info-Gap

Magic Recipe for Handling Severe Uncertainty

$$\hat{\alpha}(q,r) := \max\{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha,\tilde{u})\}, q \in \mathcal{Q}$$

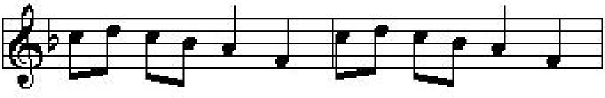
Obvious facts about this recipe

- Very popular in a number of prestigious organizations in Australia
- Afflicted by a number of fundamental flaws
- Based on a classical Voodoo Decision Theory par excellence

To understand and appreciate the flaws we have to look at

Classical Decision Theory, Robust Optimization, Voodoo Decision Theory

Classical Decision Theory



Eg.

620-262: Decision Making

A Simple Problem

Good morning Sir/Madam:

I left on your doorstep four envelopes. Each contains a sum of money. You are welcome to open any one of these envelopes and keep the money you find there.

Please note that as soon as you open an envelope, the other three will automatically self-destruct, so think carefully about which of these envelopes you should open.

To help you decide what you should do, I printed on each envelope the possible values of the amount of money (in Australian dollars) you may find in it. The amount that is actually there is equal to one of these figures.

Unfortunately the entire project is under severe uncertainty so I cannot tell you more than this.

Good luck!

Joe.

So What Do You do?

Example

Envelope	*Possible Amount (Australian dollars)*
$E1$	$20, 10, 300, 786$
$E2$	$2, 40000, 102349, 5000000, 99999999, 56435432$
$E3$	$201, 202$
$E4$	200

Vote!

Modeling and Solution

- What is a decision problem ?
- How do we model a decision problem?
- How do we solve a decision problem?

Decision Tables

Think about your problem as a table, where

- rows represents decisions
- columns represent the relevant possible states of nature
- entries represent the associated payoffs/rewards/costs

Example

Env	*Possible Amount ($AU)*				
$E1$	20	10	300	786	
$E2$	2	4000000	102349	500000000	56435432
$E3$	201	202			
$E4$	200				

Classification of Uncertainty

Classical decision theory distinguishes between three levels of uncertainty regarding the state of nature, namely

- Certainty
- Risk
- Strict Uncertainty

Terminology:

Strict Uncertainty ≡ Severe Uncertainty
≡ Ignorance
≡ True Uncertainty
≡ Knightian Uncertainty
≡ Deep
≡ Extreme
≡ Hard

Severe Uncertainty

Classical decision theory offers two basic principles for dealing with severe uncertainty, namely

- Laplace's Principle (1825)
- Wald's Principle (1945)

Conceptually:

Certainty $\overset{\text{Wald}}{\Longleftarrow}$ **Severe Uncertainty** $\overset{\text{Laplace}}{\Longrightarrow}$ Risk

$\Downarrow$

Voodoo Decision Theory
Info-Gap Decision Theory
???

Laplace's Principle of Insufficient Reason (1825)

Assume that all the states are equally likely, thus use a uniform distribution function (μ) on the state space and regard the problem as decision-making under risk.

Laplace's Decision Rule

$$\max_{d\in\mathbb{D}} \int_{s\in S(d)} r(s,d)\mu(s)\,ds \qquad \text{Continuous case}$$

$$\max_{d\in\mathbb{D}} \frac{1}{|S(d)|}\sum_{s\in S(d)} r(s,d) \qquad \text{Discrete case}$$

Wald's Maximin Principle (1945)

Inspired by Von Neumann's [1928] Maximin model for 0-sum, 2-person games: Mother Nature is and adversary and is playing against you, hence apply the worst-case scenario. This transforms the problem into a decision-making under certainty.

Wald's Maximin Rule

$$\overset{\text{You!}}{\max_{d\in\mathbb{D}}}\ \overset{\text{Mama}}{\min_{s\in S(d)}}\ f(d,s)$$

Historical perspective: William Shakespeare (1564-1616)

> *The gods to-day stand friendly, that we may,*
> *Lovers of peace, lead on our days to age!*
> *But, since the affairs of men rests still incertain,*
> *Let's reason with the worst that may befall.*
>
> *Julius Caesar, Act 5, Scene 1*

Laplace vs Wald

Example

Env	*Possible Amount ($AU)*				
$E1$	20	10	300	786	
$E2$	2	4000	102349	50000	56435
$E3$	201	202			
$E4$	200				

Example

Env	*Possible Amount ($AU)*					*Laplace*	*Wald*
$E1$	20	10	300	786		279	10
$E2$	2	4000	10234	50000	56435	24134.2	2
$E3$	201	202				201.5	201
$E4$	200					200	200

Laplace vs Wald

Example

Env	*Possible Amount ($AU)*					*Laplace*	*Wald*
$E1$	20	10	300	786		279	10
$E2$	2	4000	10234	50000	56435	24134.2	2
$E3$	201	202				201.5	201
$E4$	200					200	200

Robust Decision-Making

WIKIPEDIA

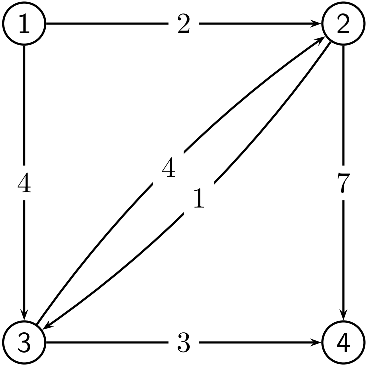
Robustness is the quality of being able to withstand stresses, pressures, or changes in procedure or circumstance. A system, organism or design may be said to be "robust" if it is capable of coping well with variations (sometimes unpredictable variations) in its operating environment with minimal damage, alteration or loss of functionality.

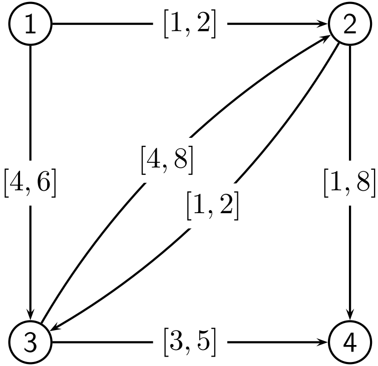
- Applies to both (known) variability and uncertainty
- Origin (in OR/MS): 1970s
- A very "hot" area of research these days ...
- See bibliography

Robust Optimization

Simple Example

Shortest path problem with variable arc lengths





Robust Decision-Making

Classification

- Robust Satisficing
  Robustness with respect to constraints of a satisficing problem or an optimization problem.
- Robust Optimizing
  Robustness with respect to the objective function of an optimization problem.
- Robust optimizing and satisficing
  Robustness with respect to both the objective function and constraints of an optimization problem.

Dominated by Wald's Maximin models and Savage's Minimax Regret models.

Robust Decision-Making

Classification

- Robust Satisficing

$$\text{Problem } P(u), u \in U: \quad \text{Find an } x \in X \text{ such that } g(x, u) \in C$$

- Robust Optimizing

$$\text{Problem } P(u), u \in U: \quad z^* := \operatorname*{opt}_{x \in X} f(x, u)$$

- Robust optimizing and satisficing

$$\text{Problem } P(u), u \in U: \quad z^* := \operatorname*{opt}_{x \in X(u)} f(x, u)$$

Robust Decision-Making

Robustness á la Maximin

- Robust Optimizing

$$z^* := \max_{d\in\mathbb{D}} \min_{s\in S(d)} f(d,s)$$

- Robust Satisficing

$$z^* : = \max_{d\in\mathbb{D}} \min_{s\in S(d)} \varphi(d,s) := \begin{cases} \beta(d) & , \quad g(d,s)\in C \\ -\infty & , \quad g(d,s)\notin C \end{cases}$$

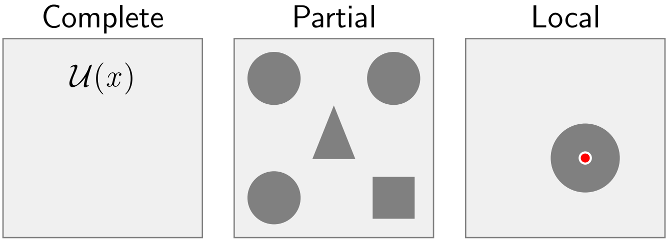
- Robust optimizing and satisficing

$$z^* : = \max_{d\in\mathbb{D}} \min_{s\in S(d)} \psi(d,s) := \begin{cases} \gamma(d,s) & , \quad g(d,s)\in C \\ -\infty & , \quad g(d,s)\notin C \end{cases}$$

Robust Decision-Making

Degree of Robustness

- Complete
  $\forall u \in \mathcal{U}(x)$ (very conservative)
- Partial
  $\forall u \in U(x) \subseteq \mathcal{U}(x)$
- Local
  $\forall u \in U(x, \tilde{u}) \subseteq \mathcal{U}(x)$



Robust Decision-Making

Robustness á la Maximin

- Complete robustness

$$z^* := \max_{d\in\mathbb{D}} \min_{s\in S(d)} f(d,s)$$

- Partial robustness

$$z^* := \max_{\substack{d\in\mathbb{D}\\ U\subseteq S(d)}} \min_{s\in U} g(d,U,s)$$

where

$$g(d,U,s) := \begin{cases} \rho(U) & , \quad f(d,s) \geq f^*(s) \\ 0 & , \quad \text{otherwise} \end{cases}$$

Robust Decision-Making

Robustness á la Maximin

Local robustness

$$z^* := \max_{\substack{d \in \mathbb{D} \\ \alpha \geq 0}} \min_{s \in U(\alpha, \tilde{s})} g(d, \alpha, s)$$

where

$$g(d, \alpha, s) := \begin{cases} \alpha & , \quad f(d, s) \geq c \\ 0 & , \quad \text{otherwise} \end{cases}$$

Remark:
This approach is not suitable for severe uncertainty.

Voodoo

Voodoo Decision Theory

Encarta online Encyclopedia

Voodoo n

1. A religion practiced throughout Caribbean countries, especially Haiti, that is a combination of Roman Catholic rituals and animistic beliefs of Dahomean enslaved laborers, involving magic communication with ancestors.
2. Somebody who practices voodoo.
3. A charm, spell, or fetish regarded by those who practice voodoo as having magical powers.
4. A belief, theory, or method that lacks sufficient evidence or proof.

Voodoo

Voodoo

Voodoo

Voodoo

Voodoo Decision Theory

Apparently very popular,

Example

The behavior of Kropotkin's cooperators is something like that of decision makers using Jeffrey expected utility model in the Max and Moritz situation. Are ground squirrels and vampires using voodoo decision theory?

Brian Skyrms
Evolution of the Social Contract
Cambridge University Press, 1996.

Issue:

Evidential dependence, but causal independence.

The legend

An old legend has it that an ancient treasure is hidden in an Asian-Pacific island.

You are in charge of the treasure hunt. How would you plan the operation?

The legend

Main issue: location, location, location!

Terminology

Certainty

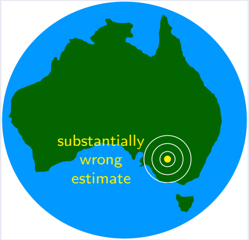
Risk

Severe Uncertainty

Voodooism

The Fundamental Theorem of Voodoo Decision Making

$\approx$



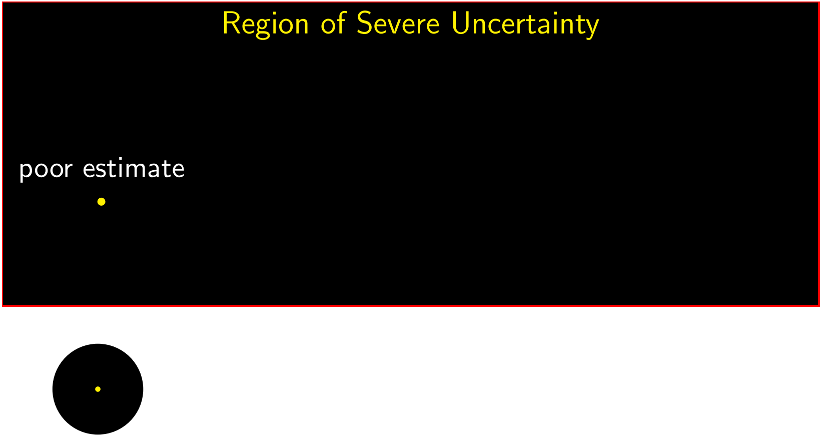
Severe Uncertainty

1·2·3 Recipe

1. Ignore the severe uncertainty.
2. Focus on the substantially wrong estimate you have.
3. Conduct the analysis in the immediate neighborhood of this estimate.

Voodooism

Voodoo Decision-Making



Voodooism

Voodoo Decision-Making

Just in case, . . . , the difficulty is that

Under SEVERE uncertainty

The estimate we use is

- A wild guess.
- A poor indication of the true value.
- Likely to be substantially wrong.

Hence,

Beware!

Results obtained in the neighborhood of the estimate are likely to be substantially wrong in the neighborhood of the true value.

Voodooism

The Curse of Preference Reversal

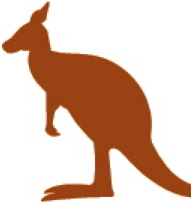
Region of Severe Uncertainty

poor estimate

Plan A is great!!
Plan B is a lemon!!

Plan A is a lemon!!
Plan B is great!!

true value



VS

Voodooism

Summary

GI → Model → GO

Wrong → Model → Wrong

Question

What is the most popular Voodoo Decision Theory for robust decision-making under severe uncertainty in a number of prestigious organizations in

Australia

?

Info-Gap Revisited

Impressive Self-Portrait

Info-gap decision theory is radically different from all current theories of decision under uncertainty. The difference originates in the modelling of uncertainty as an information gap rather than as a probability. The need for info-gap modeling and management of uncertainty arises in dealing with severe lack of information and highly unstructured uncertainty.

Ben-Haim [2006, p. xii]

In this book we concentrate on the fairly new concept of information-gap uncertainty, whose differences from more classical approaches to uncertainty are real and deep.

Ben-Haim [2006, p. 11]

Info-Gap

Obvious Questions

1. Does Info-Gap substantiate these very strong claims?
2. Are these claims valid?

Not So Obvious Answers

1. No, it does not.
2. Certainly not.

It is therefore important to subject Info-Gap to a formal analysis – that actually should have been done seven years ago:

Info-Gap
Formal vs Analysis
Classical Decision Theory

Good news: should take no more than 5-10 minutes!

Info-Gap

Meaning of Severe Uncertainty

- The region of uncertainty is usually relatively large, often unbounded.
- The uncertainty cannot be quantified by a probabilistic model.
- If there is an estimate of the parameter of interest, then the estimate is
  - A wild guess
  - A poor indication of the true value
  - Likely to be substantially wrong

Info-Gap

Meaning of Severe Uncertainty

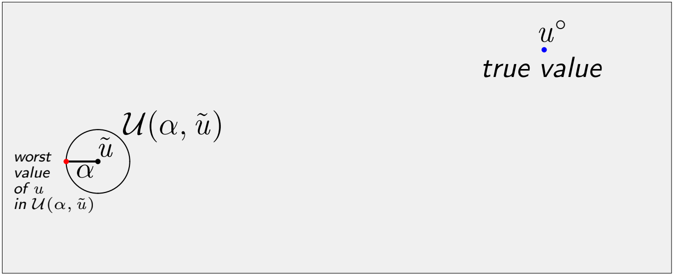
bio-security homeland-security

Generic Info-Gap Model

Complete Model

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \quad (1)$$

Region of Severe Uncertainty, $\mathfrak{U}$



Info-Gap

Complete Generic Model

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \qquad (2)$$

Fundamental FAQs

1. Is this new? — Definitely not!
2. Is this radically different? — Definitely not!
3. Does it make sense? — Definitely not!

So what is all this hype about Info-Gap ?!

Good question!

Info-Gap

First Impression

Complete Generic Model

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \tag{3}$$

Observations

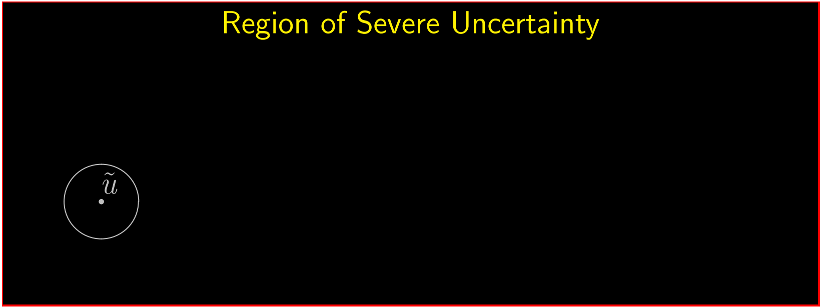
- This model does not deal with severe uncertainty, it simply and unceremoniously ignores it.
- The analysis is invariant with $\mathfrak{U}$: the same solution for all $\mathfrak{U}$ such that $\mathcal{U}(\hat{\alpha}(r_c), \tilde{u}) \subseteq \mathfrak{U}$.
- This model is fundamentally flawed.
- This model advocates voodoo decision-making.

Info-Gap

First Impression

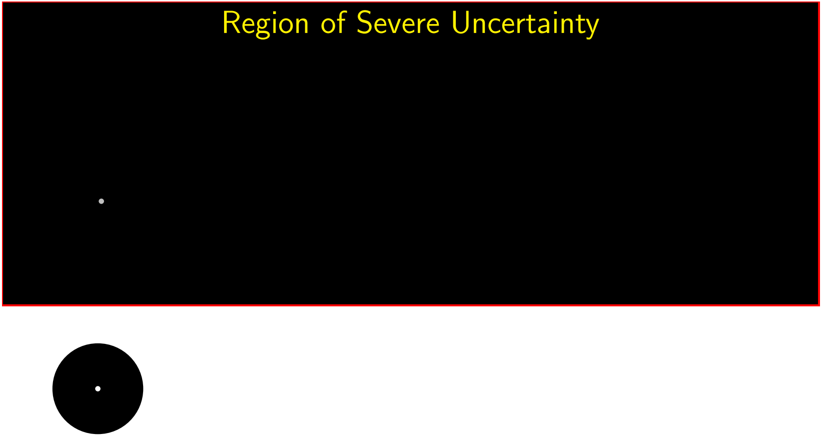
Fool-Proof Recipe

Step 1: *Ignore the severe uncertainty.*
Step 2: *Focus instead on the poor estimate and its immediate neighborhood.*



Info-Gap

First Impression



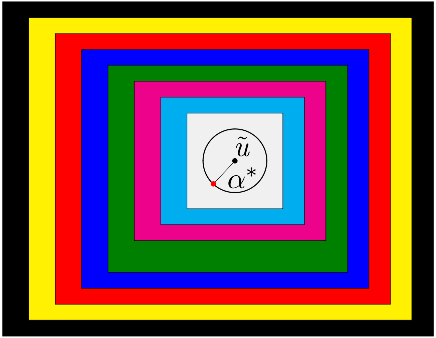
Recall that this is voodoo decision making!

Info-Gap

Complete Generic Model

$$\alpha^* := \max_{q \in \mathbb{Q}} \ \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q, u) \right\} \tag{4}$$

Fundamental Flaw



Info-Gap

More formally

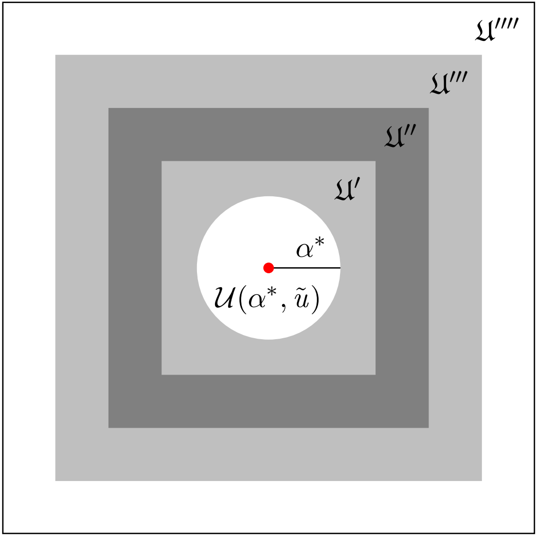
Theorem (Sniedovich, 2007)

Info-Gap's robustness model is invariant to the size of the total region of uncertainty $\mathfrak{U}$ for all $\mathfrak{U}$ larger than $\mathcal{U}(\alpha^*, \tilde{u})$, where $\alpha^* := \hat{\alpha}(r_c)$.
That is, the model yields the same results for all $\mathfrak{U}$ such that

$$\mathcal{U}(\alpha^* + \varepsilon, \tilde{u}) \subseteq \mathfrak{U} \ , \ \varepsilon > o$$

Info-Gap

Info-Gap's Invariance Property



Info-Gap

Theorem (Sniedovich 2007, 2008)

Info-Gap's robustness model is a simple instance of Wald's Maximin model. Specifically,

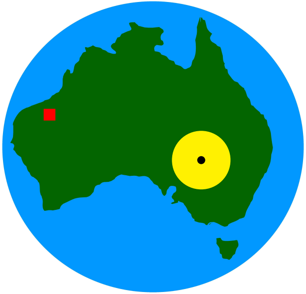
$$\begin{aligned}\alpha(q) &:= \max_{\alpha\geq 0}\left\{\alpha : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right\}\\ &= \max_{\alpha\geq 0}\ \min_{u\in\mathcal{U}(\alpha,\tilde{u})} \psi(q,\alpha,u)\end{aligned}$$

where

$$\psi(q,\alpha,u) := \begin{cases}\alpha\ , & r_c \leq R(q,u)\\ 0\ , & r_c > R(q,u)\end{cases}\ ,\alpha \geq 0, q\in\mathbb{Q}, u\in\mathcal{U}(\alpha,\tilde{u})$$

Info-Gap: Typical misconception

Treasure Hunt



- Region of uncertainty.
- Estimate of the location.
- Region affecting Info-Gap's analysis.
- True (unknown) location.

Hence, Info-gap may conduct its robustness analysis in the vicinity of Brisbane (QLD), whereas for all we know the true location of the treasure may be somewhere in the middle of the Simpson desert or perhaps in down town Melbourne (VIC). Perhaps.

Conclusions

- Decision-making under severe uncertainty is difficult.
- It is a thriving area of research/practice.
- The Robust Optimization literature is extremely relevant.
- The Decision Theory literature is extremely relevant.
- The Operations Research literature is very relevant.
- Info-Gap's decision model is neither new nor radically different.
- Info-Gap's uncertainty model is fundamentally flawed and unsuitable for decision-making under severe uncertainty.
- Info-Gap exhibits a severe information-gap about the state of the art in decision-making under severe uncertainty.
- It is time to reassess the use of Info-Gap in Australia.
- Join the Campaign
- Join the Research

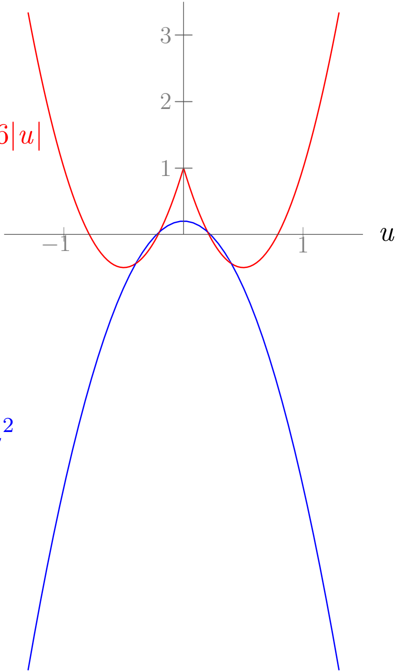
Off the record

The Ten Natural Laws of Operations Analysis

1. Ignore the problem and go immediately to the solution, that is where the profit lies.
2. There are no small problems only small budgets.
3. Names are control variables.
4. Clarity of presentation leads to aptness of critique.
5. Invention of the wheel is always on the direct path of a cost plus contract.
6. Undesirable results stem only from bad analysis.
7. It is better to extend an error than to admit to a mistake.
8. Progress is a function of the assumed reference system.
9. Rigorous solutions to assumed problems are easier to sell than assumed solutions to rigorous problems.
10. In desperation address the problem.

Bob Bedow, *Interfaces* 7(3), p. 122, 1979.

$$R(q', u) = 1 + 6u^2 - 6|u|$$



$$R(q'', u) = 0.2 - 4u^2$$


Moresby
Darwin
Northern Territory
Queensland
Australia
Western Australia
Brisbane
South Australia
Perth
New South Wales
Sydney
Adelaide
Canberra
ACT
Victoria
Melbourne
Tasmania
Hobart

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