The Art and Science of Robust Decision-Making

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AIRO 2008
September 8-11, 2008
Ischia, Italy

Program

- How do you make robust decisions in the face of severe uncertainty?
    - Classical decision theory
    - Robust decision-making
    - Voodoo decision theory
    - My Maximin (and related) "Campaigns"
    - Info-gap decision theory

Admin

This is a

Math Classification G

presentation.

Math Classification MA +18

versions can be found at

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AU Perspective

New Secret Weapon Against Severe Uncertainty

$$\hat{\alpha}(q) := \max\{\alpha \geq 0 : r \leq R(q,u), \forall u \in U(\alpha, \tilde{u})\}, q \in \mathcal{Q}$$

Known as

Info-Gap Robustness Model

Ben-Haim (1996, 2001, 2006)

Very popular in a number of research organizations in Australia

Classification of Uncertainty

Classical decision theory distinguishes between three levels of uncertainty regarding the state of nature, namely

- Certainty
- Risk
- Strict Uncertainty

Terminology:

Strict Uncertainty ≡ Severe Uncertainty
≡ Ignorance
≡ True Uncertainty
≡ Knightian Uncertainty
≡ Deep
≡ Extreme
≡ Hard
≡ Fundamental

Severe Uncertainty

Classical decision theory offers two basic principles for dealing with severe uncertainty, namely

- Laplace's Principle (1825)
- Wald's Principle (1945)

Conceptually:

$$\text{Certainty} \overset{\text{Wald}}{\Longleftarrow} \boxed{\text{Severe Uncertainty}} \overset{\text{Laplace}}{\Longrightarrow} \text{Risk}$$

$$\Downarrow$$

Voodoo Decision Theory
Info-Gap Decision Theory
???

Laplace's Principle of Insufficient Reason (1825)

Assume that all the states are equally likely, thus use a uniform distribution function (μ) on the state space and regard the problem as decision-making under risk.

Laplace's Decision Rule

$$\max_{d \in \mathbb{D}} \int_{s \in S(d)} r(s, d)\mu(s)\, ds \qquad \text{Continuous case}$$

$$\max_{d \in \mathbb{D}} \frac{1}{|S(d)|} \sum_{s \in S(d)} r(s, d) \qquad \text{Discrete case}$$

Wald's Maximin Principle (1945)

Inspired by Von Neumann's [1928] Maximin model for 0-sum, 2-person games: Mother Nature is and adversary and is playing against you, hence apply the worst-case scenario. This transforms the problem into a decision-making under certainty.

Nice Plain Language Formulation

The maximin rule tells us to rank alternatives by their worst possible outcomes: we are to adopt the alternative the worst outcome of which is superior to the worst outcome of the others.

Rawls, J., *Theory of Justice*, 1971, p. 152

Wald's Maximin Principle (1945)

Historical perspective

The gods to-day stand friendly, that we may,
Lovers of peace, lead on our days to age!
But, since the affairs of men rests still incertain,
Let's reason with the worst that may befall.

William Shakespeare (1564-1616)
Julius Caesar, Act 5, Scene 1

Classic Format

$$\underset{d\in\mathbb{D}}{\overset{\text{You!}}{\max}} \quad \underset{s\in S(d)}{\overset{\text{Mama}}{\min}} \quad f(d,s)$$

About Maximin/Minimax formulations

Classical Format

$$\underset{d\in\mathbb{D}}{\overset{\text{You!}}{\max}} \ \underset{s\in S(d)}{\overset{\text{Mama}}{\min}} \ f(d,s)$$

Mathematical Programming Format

$$\underset{\substack{d\in\mathbb{D}\\ v\in\mathbb{R}}}{\overset{\text{You!}}{\max}} \left\{ v : f(d,s) \geq v \ , \ \overset{\downarrow \ \text{Mama}}{\forall s \in S(d)} \right\}$$

Note: if $S(d)$ is "continuous", then this is a semi-infinite program.

Robust Decision-Making

WIKIPEDIA

Robustness is the quality of being able to withstand stresses, pressures, or changes in procedure or circumstance. A system, organism or design may be said to be "robust" if it is capable of coping well with variations (sometimes unpredictable variations) in its operating environment with minimal damage, alteration or loss of functionality.

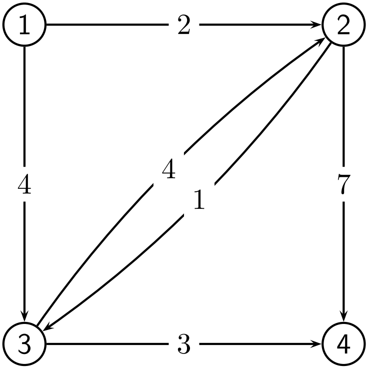
- Applies to both (known) variability and uncertainty
- Origin: probably late 1920's (game theory).
- In OR and Optimization: late 1960s early 1970s.
- Major difficulty: solution procedures.
- A very "hot" area of research these days ...
- See bibliography

Robust Optimization

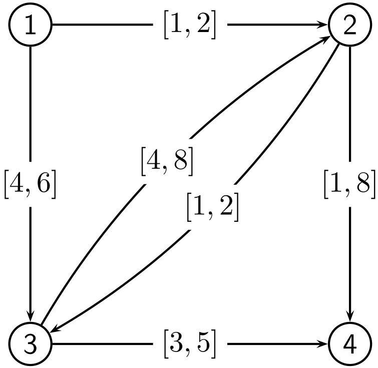
Simple Example

Shortest path problem with variable arc lengths

"Conventional version"



“Robust version”



Robust Decision-Making

Role of Maximin/Minimax in Robustness Analysis

But as we defined robustness to mean insensitivity with regard to small deviations from assumptions, any quantitative measure of robustness must somehow be concerned with the maximum degradation of performance possible for an ϵ-deviation from the assumptions. The optimally robust procedure minimizes this degradation and hence will be a minimax procedure of some kind.

Huber (1981, pp. 16-17)

Experience: Modeling aspects can be subtle!

- Optimizing vs Satisficing
- Complete vs Partial vs Local
- (Mis) Interpretation

Robust Decision-Making

Classification

- Robust Satisficing (eg. Soyster (1973), Ben-Tal and Nemirovski (1999))
  Robustness with respect to constraints of a satisficing problem or an optimization problem.
- Robust Optimizing (eg. classical Maximin/Minimax)
  Robustness with respect to the objective function of an optimization problem.
- Robust Optimizing and Satisficing (eg. Ben-Tal and Nemirovski (2002))
  Robustness with respect to both the objective function and constraints of an optimization problem.

Robust Decision-Making

Classification

- Robust Satisficing

  Problem $P(u), u \in U$:

  Find an $x \in X$ such that $g(x, u) \in C$

- Robust Optimizing

  Problem $P(u), u \in U$:

  $$z^* := \operatorname*{opt}_{x \in X} f(x, u)$$

- Robust Optimizing and Satisficing

  Problem $P(u), u \in U$:

  $$z^* := \operatorname*{opt}_{x \in X(u)} f(x, u)$$

Robust Decision-Making

Robustness á la Maximin

- Robust Optimizing (Classical Maximin (1945))

$$\max_{d\in\mathbb{D}} \min_{s\in S(d)} f(d,s) \equiv \max_{\substack{d\in\mathbb{D}\\ v\in\mathbb{R}}} \{v : f(d,s) \geq v, \forall s \in S(d)\}$$

- Robust Satisficing (eg. Soyster (1973), Ben-Tal and Nemirovski (1999))

$$\max_{d\in\mathbb{D}} \{\beta(d) : g(d,s) \in C, \forall s \in S(d)\} \equiv \max_{d\in\mathbb{D}} \min_{s\in S(d)} \varphi(d,s)$$

$$\varphi(d,s) := \begin{cases} \beta(d) & , \quad g(d,s) \in C \\ -\infty & , \quad g(d,s) \notin C \end{cases}$$

Robust Decision-Making

Robustness á la Maximin

- Robust Optimizing and Satisficing (eg. Ben-Tal and Nemirovski (2002))

$$\max_{\substack{d\in D\\ v\in\mathbb{R}}} \{v : \gamma(d,s) \geq v, g(d,s) \in C, \forall s \in S(d)\}$$
$$\equiv \max_{d\in\mathbb{D}} \min_{s\in S(d)} \psi(d,s)$$

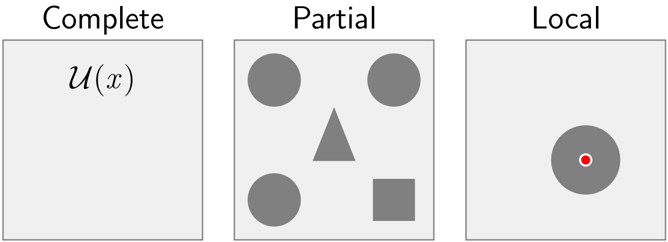
where

$$\psi(d,s) := \begin{cases} \gamma(d,s) & , \quad g(d,s) \in C \\ -\infty & , \quad g(d,s) \notin C \end{cases}$$

Robust Decision-Making

Degree of Robustness

- Complete (conventional)
  $\forall u \in \mathcal{U}(x)$ (very conservative)
- Partial (eg. Starr (1962), Schneller and Sphicas (1983))
  $\forall u \in U(x) \subseteq \mathcal{U}(x)$
- Local (eg. Ben-Haim (2001, 2006, 2008))
  $\forall u \in U(x,\tilde{u}) \subseteq \mathcal{U}(x) \quad (U(x,\tilde{u}) =$ neighborhood of $\tilde{u}$ $)$



Robust Decision-Making

Robustness á la Maximin

Complete robustness

$$z^* := \max_{d\in\mathbb{D}} \min_{s\in S(d)} f(d,s)$$

$$= \max_{\substack{d\in\mathbb{D}\\ v\in\mathbb{R}}} \{v : f(d,s) \geq v, \forall s \in S(d)\}$$

Robust Decision-Making

Robustness á la Maximin

Partial robustness

$$\begin{aligned} \rho(U) &= \text{“size” of set } U \\ z^* := &\max_{\substack{d\in\mathbb{D}\\ U\subseteq S(d)}} \{\rho(U) : f(d,s) \in C(d,s), \forall s \in U\} \\ &= \max_{\substack{d\in\mathbb{D}\\ U\subseteq S(d)}} \min_{s\in U} g(d,U,s) \end{aligned}$$

where

$$g(d,U,s) := \begin{cases} \rho(U) & , \quad f(d,s) \in C(d,s) \\ 0 & , \quad \text{otherwise} \end{cases}$$

Robust Decision-Making

Robustness á la Maximin

Local robustness

$$U(d,\alpha,\tilde{s}) = \text{ neighborhood of "size" } \alpha \text{ around } \tilde{s}$$

$$\alpha^* := \max_{\substack{d\in\mathbb{D}\\ \alpha\geq 0}} \{\alpha : f(d,s) \in C(d,s), \forall s \in U(d,\alpha,\tilde{s})\}$$

$$= \max_{\substack{d\in\mathbb{D}\\ \alpha\geq 0}} \min_{s\in U(d,\alpha,\tilde{s})} g(d,\alpha,s)$$

$$g(d,\alpha,s) := \begin{cases} \alpha & , \ f(d,s) \in C(d,s) \\ -\infty & , \ \text{otherwise} \end{cases}$$

Remark:
This model is local in nature, hence is unsuitable for severe uncertainty.

Voodoo Decision Theory

Voodoo Decision Theory

Encarta online Encyclopedia

Voodoo n

1. A religion practiced throughout Caribbean countries, especially Haiti, that is a combination of Roman Catholic rituals and animistic beliefs of Dahomean enslaved laborers, involving magic communication with ancestors.
2. Somebody who practices voodoo.
3. A charm, spell, or fetish regarded by those who practice voodoo as having magical powers.
4. A belief, theory, or method that lacks sufficient evidence or proof.

Voodoo Decision Theory

Apparently very popular,

Example

The behavior of Kropotkin's cooperators is something like that of decision makers using Jeffrey expected utility model in the Max and Moritz situation. Are ground squirrels and vampires using voodoo decision theory?

Brian Skyrms
Evolution of the Social Contract
Cambridge University Press, 1996.

Issue:

Evidential dependence, but causal independence.

The legend

An old legend has it that an ancient treasure is hidden in an Asian-Pacific island.

You are in charge of the treasure hunt. How would you plan the operation?

The legend

Main issue: location, location, location!

Terminology

Certainty

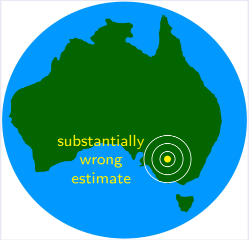
Risk

Severe Uncertainty

Voodooism

The Fundamental Theorem of Voodoo Decision Making

$\approx$



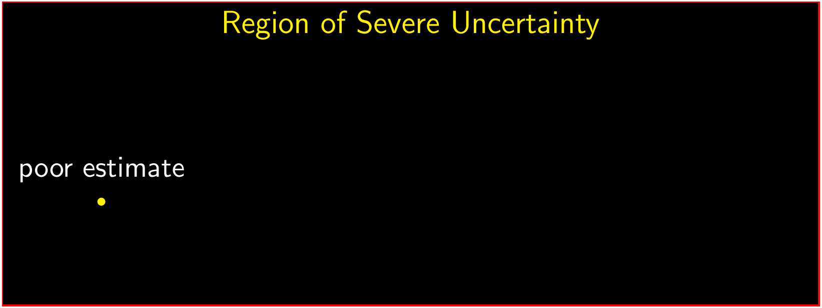
Severe Uncertainty

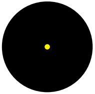
1·2·3 Recipe

1. Ignore the severe uncertainty.
2. Focus on the substantially wrong estimate you have.
3. Conduct the analysis in the immediate neighborhood of this estimate.

Voodooism

Voodoo Decision-Making





Voodooism

Conventional Decision Theory

$$\text{GI} \longrightarrow \boxed{\mathfrak{Model}} \longrightarrow \text{GO}$$

$$\text{Wrong} \longrightarrow \boxed{\mathfrak{Model}} \longrightarrow \text{Wrong}$$

The robustness of any decision and the risk incurred in making that decision is only as good as the estimates on which it is based. Making estimation even more challenging, virtually all estimates that affect decisions are uncertain. Uncertainty can not be eliminated, but it can be managed.

Top Ten Challenges for Making Robust Decisions
The Decision Expert Newsletter, Volume 1; Issue 2
`http://www.robustdecisions.com/newsletter0102.php`

Voodooism

V[illegible] Decisio[illegible]ory

$$\text{GI} \longrightarrow \boxed{[\text{illegible}]} \longrightarrow \text{GO}$$

$$\text{Wrong} \longrightarrow \boxed{[\text{illegible}]} \longrightarrow \text{Wrong}$$

Voodooism

Voodoo Decision Theory

$$\underset{GI}{\textit{garbage}} \longrightarrow \boxed{\mathfrak{Model}} \longrightarrow \underset{GO}{\textbf{gold}}$$

$$\textbf{Wrong} \longrightarrow \boxed{\mathfrak{Model}} \longrightarrow \textbf{Right}$$

Alchemy

Info-Gap Revisited

Impressive Self-Portrait

Info-gap decision theory is radically different from all current theories of decision under uncertainty. The difference originates in the modelling of uncertainty as an information gap rather than as a probability. The need for info-gap modeling and management of uncertainty arises in dealing with severe lack of information and highly unstructured uncertainty.

Ben-Haim [2006, p. xii]

In this book we concentrate on the fairly new concept of information-gap uncertainty, whose differences from more classical approaches to uncertainty are real and deep.

Ben-Haim [2006, p. 11]

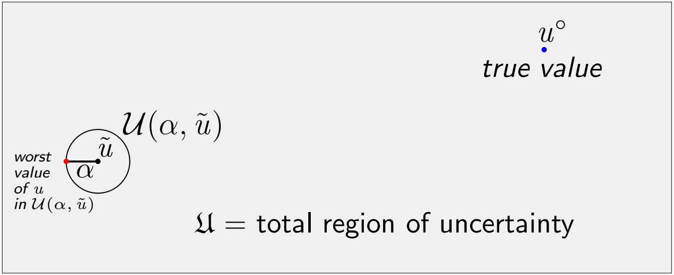
Info-Gap Decision Theory

Complete Generic Robustness Model

$$\hat{\alpha}(r_c) := \max_{q \in \mathbb{Q}} \max \left\{ \alpha \geq 0 : r_c \leq \min_{u \in \mathcal{U}(\alpha, \tilde{u})} R(q,u) \right\}$$

$$\mathcal{U}(\alpha, \tilde{u}) \subseteq \mathcal{U}(\alpha + \varepsilon, \tilde{u}), \forall \varepsilon > 0$$

Region of Severe Uncertainty, $\mathfrak{U}$



Info-Gap Decision Theory

Complete Generic Robustness Model

$$\hat{\alpha}(r_c) := \max_{q\in\mathbb{Q}} \ \max \left\{ \alpha \geq 0 : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u) \right\}$$

Fundamental FAQs

1. Is this new? Definitely not!
2. Is this radically different? Definitely not!
3. Does it make sense? Definitely not!

So what is all this hype about Info-Gap ?!

Good question!

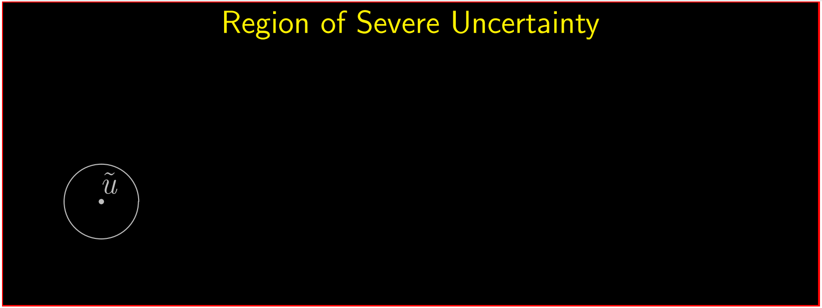
Info-Gap Decision Theory

First Impression

Fool-Proof Recipe

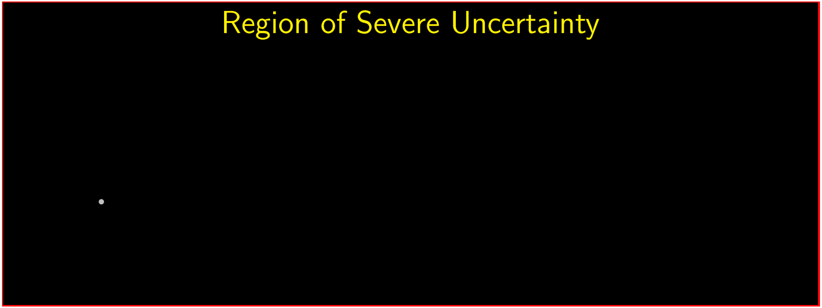
Step 1: *Ignore the severe uncertainty.*

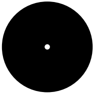
Step 2: *Focus instead on the poor estimate and its immediate neighborhood.*



Info-Gap Decision Theory

First Impression





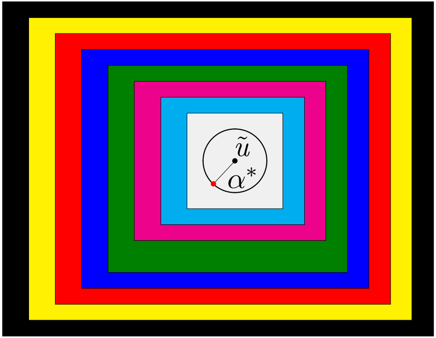
Recall that this is voodoo decision making!

Info-Gap Decision Theory

Complete Generic Robustness Model

$$\alpha^* := \max_{q\in\mathbb{Q}} \max\left\{\alpha \geq 0 : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right\}$$

Fundamental Flaw



Info-Gap Decision Theory

More formally

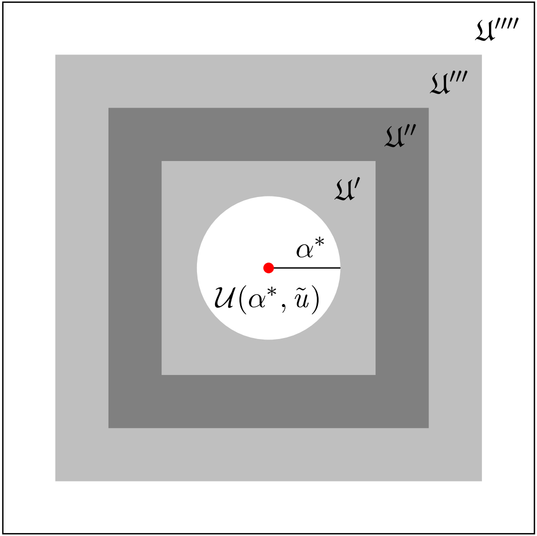
Invariance Theorem (Sniedovich, 2007)

Info-Gap's robustness model is invariant to the size of the total region of uncertainty $\mathfrak{U}$ for all $\mathfrak{U}$ larger than $\mathcal{U}(\alpha^*, \tilde{u})$, where $\alpha^* := \hat{\alpha}(r_c)$.
That is, the model yields the same results for all $\mathfrak{U}$ such that

$$\mathcal{U}(\alpha^* + \varepsilon, \tilde{u}) \subseteq \mathfrak{U} \ , \ \ \varepsilon > o$$

Info-Gap Decision Theory

Info-Gap's Invariance Property



Info-Gap Decision Theory

Maximin Theorem (Sniedovich 2007, 2008)

Info-Gap's robustness model is a simple instance of Wald's Maximin model. Specifically,

$$\begin{aligned}\alpha(q) &:= \max_{\alpha\geq 0}\left\{\alpha : r_c \leq \min_{u\in\mathcal{U}(\alpha,\tilde{u})} R(q,u)\right\}, \quad q\in\mathbb{Q}\\ &= \max_{\alpha\geq 0}\min_{u\in\mathcal{U}(\alpha,\tilde{u})} \psi(q,\alpha,u)\end{aligned}$$

where

$$\psi(q,\alpha,u) := \begin{cases}\alpha, & r_c \leq R(q,u)\\ 0, & r_c > R(q,u)\end{cases}, \alpha\geq 0, q\in\mathbb{Q}, u\in\mathcal{U}(\alpha,\tilde{u})$$

Conclusions

- Decision-making under severe uncertainty is difficult.
- It is a thriving area of research/practice.
- The Robust Optimization literature is extremely relevant.
- The Decision Theory literature is extremely relevant.
- The Operations Research literature is very relevant.
- Info-Gap's robustness model is neither new nor radically different.
- Info-Gap's uncertainty model is fundamentally flawed and is unsuitable for decision-making under severe uncertainty.
- Info-Gap Decision Theory exhibits a severe information-gap about the state of the art in decision-making under severe uncertainty.

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